SECOND INTERNATIONAL MEETING OF THE ASSOCIATION FOR THE PHILOSOPHY OF MATHEMATICAL PRACTICE

UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN
OCTOBER 3–4, 2013

All talks will take place on the second floor of the Levis Faculty Center.

October 3

9:00–10:15am Main room	Marcus Giaquinto, Department of Philosophy, University College London: "Epistemic roles of visual experience in mathematical knowledge acquisition"
10:15–11:30am Main room	Elaine Landry, Department of Philosophy, University of California, Davis: "Plato was *Not* a Mathematical Platonist"
11:30–12:00pm	Break
12:00–12:35pm Main room	Dirk Schlimm, Department of Philosophy, McGill University: "The early development of Dedekind's notion of mapping"
12:00–12:35pm Music room	Joachim Frans, Centre for Logic and Philosophy of Science, VUB Brussels and Erik Weber, Centre for Logic and Philosophy of Science, Ghent University: "Mechanistic Explanation And Explanatory Proofs In Mathematics"
12:35–1:10pm Main room	Emmylou Haffner, Department of History and Philosophy of Science, Université Paris Diderot - Paris 7: "Arithmetization of mathematics, the Dedekindian way"
12:35–1:10pm Music room	Andrew Aberdein, Department of Humanities and Communication, Florida Institute of Technology and Matthew Inglis, Mathematics Education Centre, Loughborough University: "Explanation and Explication In Mathematical Practice"
1:10–2:30pm	Lunch break
2:30–3:05pm Main room	John Baldwin, Department of Mathematics, Statistics and Computer Science, University of Illinois at Chicago: "From Geometry to Algebra and Analysis"
2:30–3:05pm Music room	Ashton Sperry-Taylor, Department of Philosophy, University of Missouri: "Modeling The State Of Nature"
3:05–3:40pm Main room	Sylvain Cabanacq, Department of History and Philosophy of Science, Université Paris Diderot - Paris 7: "Contrasting proofs: categorical tools, model-theoretical methods and the Manin-Mumford Conjecture"
3:05–3:40pm Music room	Luca San Mauro, Center of Philosophy, Scuola Normale Superiore, Pisa: "Algorithms, formalization and exactness: a philosophical study of the practical use of Church- Turing Thesis"
3:40–4:15pm	Break
4:15–5:30pm Main room	Chris Pincock, Department of Philosophy, The Ohio State University: "Felix Klein as a Prototype for the Philosophy of Mathematical Practice"
	A meeting of APMP members follows.

October 4

9:00–10:15am Main room	Colin McLarty, Department of Philosophy, Case Western Reserve University: "Proofs in practice"
10:15–10:50am Main room	Ken Manders, Department of Philosophy, University of Pittsburgh: "Problems and Prospects for Philosophy of Mathematical Understanding"
10:15–10:50am Music room	Ramzi Kebaili, Department of History and Philosophy of Science, Université Paris Diderot - Paris 7: "Examples of 'synthetic a priori' statements in mathematical practice"
10:50–11:20am	Break
11:20–11:55am Main room	Susan Vineberg, Department of Philosophy, Wayne State University: "Are There Objective Facts of Mathematical Depth?"
11:20–11:55am Music room	Oran Magal, Department of Philosophy, McGill University: "Teratological Investigations: What's in a monster?"
11:55–12:30pm Main room	Philip Ehrlich, Department of Philosophy, Ohio University: "A Re-examination of Zeno's Paradox of Extension"
11:55–12:30pm Music room	Madeline Muntersbjorn, Department of Philosophy, University of Toledo: "Cognitive Diversity and Mathematical Progress"
12:30–2:00pm	Lunch break
2:00–2:35pm Main room	Michele Friend, Department of Philosophy, George Washington University: "Using a Paraconsistent Formal Theory of Logic Metaphorically"
2:00–2:35pm Music room	José Ferreirós, Department of Philosophy and Logic, University of Seville and Elías Fuentes Guillén, Department of Philosophy, Logic and Aesthetics, University of Salamanca: "Bolzano's 'Rein analytischer Beweis…' reconsidered"
2:35–3:10pm Main room	Bernd Buldt, Department of Philosophy, Indiana University-Purdue University Fort Wayne: "Mathematics—Some "Practical" Insights"
2:35–3:10pm Music room	Jacobo Asse Dayán, Program in Philosophy of Science, Universidad Nacional Autónoma de México: "The Intentionality and Materiality of Mathematical Objects"
3:10–3:40pm	Break
3:40–4:15pm Main room	Rochelle Gutiérrez, Department of Curriculum and Instruction, University of Illinois at Urbana-Champaign, "What is Mathematics? The Roles of Ethnomathematics and Critical Mathematics in (Re)Defining Mathematics for the Field of Education"
3:40–4:15pm Music room	Danielle Macbeth, Department of Philosophy, Haverford College: "Rigor, Deduction, and Knowledge in the Practice of Mathematics"
4:15–5:30pm Main room	Marco Panza, Institute of History and Philosophy of Science and Technology, Université Paris 1 Panthéon-Sorbonne: "On the Epistemic Economy of Formal Definitions"
	The conference dinner follows at 7pm at the ACES Library, Heritage Room.

Abstracts

October 3

9:00–10:15am Marcus Giaquinto, Department of Philosophy, University College London: "Epistemic roles of visual experience in mathematical knowledge acquisition"

It is widely believed that all mathematical knowledge is a priori and therefore that visual experience has no epistemically significant role in mathematics, only an enabling or facilitating role. I will argue against that view. Attention to specific examples shows that visual experience can play a significant role in both a posteriori and a priori ways of acquiring mathematical knowledge.

10:15–11:30am Elaine Landry, Department of Philosophy, University of California, Davis: "Plato was *Not* a Mathematical Platonist"

I will argue that Plato was *not* a mathematical Platonist. My arguments will be based primarily on the evidence found in the *Republic's* Divided Line metaphor. Typically, the mathematical Platonist story is told on the basis of two realist components: a) that mathematical objects, like Platonic Forms, exist independently of us in some metaphysical realm and "the way things are" in this realm fixes the truth of mathematical statements, so that b) we come to know such truths by, somehow or other, "recollecting" how things are in the metaphysical realm. Against b), I have demonstrated, in another paper (Landry [2012]), that recollection, in the *Meno*, is not offered as a method for mathematical knowledge. What is offered as the mathematician's method for knowledge, in the *Meno* and in the *Republic*, is the *hypothetical* method. My aim in *this* paper will be to argue against a) by showing, especially given that recollection is *not* used at all in the Republic, that since both the *method* and the *faculty* used by the mathematician are distinct from those of the philosopher, then so too must be their objects. Again, taking my evidence from the Divided Line metaphor, I argue that mathematical objects are not Forms, and so do not either exist independently of us in some metaphysical realm or fix the truth of mathematical statements. Against the standard interpretation of mathematical objects as Forms, my argument will be that from both an *epistemic* and *ontic* perspective, mathematical objects and Forms must be distinct.

12:00–12:35pm Dirk Schlimm, Department of Philosophy, McGill University: "The early development of Dedekind's notion of mapping"

The notion of mapping (Abbildung) that is presented in Dedekind's "Was sind und was sollen die Zahlen?" (1888) and which is central to Dedekind's mature mathematical and philosophical outlook, did not suddenly appear fully-formed in 1888, but is the result of a continuous development that can be traced back to the earliest writings of Dedekind, namely his "Habilitationsrede" (1854) and the lecture notes on group theory and algebra (1855-58). The present contribution (based on joint work with Wilfried Sieg) aims at presenting and discussing this development with particular attention to Dedekind's work on the real numbers, "Stetigkeit und irrationale Zahlen" (1872), algebraic number theory (1863, 1871, 1877, and 1879), various drafts for booklet on the natural numbers (1872-78), and the correspondence with Cantor.

To distinguish the different conceptions of mappings that can be identified in Dedekind's writings it is useful to look at the elements that can be used as domain and range of functions and mappings. A careful look at his writings reveals that Dedekind gradually arrived at a rigorous concept of mapping that allows for different kinds of objects to be mapped to each other. Moving away from considering only numbers as possible domains and ranges for functions, Dedekind mentions correspondences between different kinds of objects in 1872, but the first time that Dedekind speaks of a mapping between different kinds of objects in only in 1888.

We also notice a change in how Dedekind treats functions and mappings a genuine objects of investigation. Despite using homomorphisms implicitly in his early algebraic notes, it was only in 1877, i.e., when "Stetigkeit und irrationale Zahlen" (1872) was already written, but before the publication of the axiomatic presentation of the natural numbers (1888), that Dedekind discussed for the first time in print the properties of mappings and explicitly formulated the properties of injectivity and surjectivity. Thus, Dedekind's draft of "Was sind und was sollen die Zahlen?" from 1872-78 is the first evidence for the development of the conceptual apparatus that is needed in order to formulate the idea that two models of an axiom system that belong to different domains of objects have the same structure (i.e., that they are isomorphic). This is crucial for an interpretation of Dedekind's work on the real numbers as being "axiomatic" (as is his later work on the natural numbers), we can now explain the lack of a categoricity theorem for the real numbers in Dedekind's 1872 publication. Moreover, this paper shows that the mathematical background of Dedekind's structuralist philosophy of mathematics emerged only gradually in his writings.

12:00–12:35pm Joachim Frans, Centre for Logic and Philosophy of Science, VUB Brussels and Erik Weber, Centre for Logic and Philosophy of Science, Ghent University: "Mechanistic Explanation And Explanatory Proofs In Mathematics"

Although there is a general consensus among philosophers of mathematics and mathematicians that mathematical explanations exist, only a few attempts have been made in the literature to formulate accounts of explanation in mathematics. This contrasts sharply with the amount of literature on scientific explanation. Following one of the dominant views on explanation, that it increases our understanding of a phenomenon by answering an explanation-seeking question, mathematical proofs can be seen as providing information in order to answer such a question. Consequently, the issue is to see what counts as an explanation-seeking question in mathematics and how satisfactory answers can be identified.

We will argue that the literature on mathematical explanation only addresses a limited kind of explanation-seeking questions, focusing on the unificatory power of mathematical proofs. Starting from the model of mechanistic explanation approach in science, we aim at further completing the theory of explanation in mathematics. In contrast with the unificationist approach to explanation, where the explanandum is subsumed under some general principles, the mechanistic approach involves the description of a collection of entities and activities that are organized such that they realize a capacity of the system. Using the case study of a proof of the butterfly theorem in geometry, we argue that the mathematical proof similarly depends upon the identification of entities, properties and difference-makers. This approach allows one to answer what-if-things-had-been-different-questions and gain insight in why the theorem is true.

12:35–1:10pm Emmylou Haffner, Department of History and Philosophy of Science, Université Paris Diderot - Paris 7: "Arithmetization of mathematics, the Dedekindian way"

In his famous 1895 address to the Royal Academy of Sciences of Göttingen, Felix Klein presented what he called "Arithmetisierung der Mathematik" (the arithmetizing of mathematics), a phrase borrowed to Kronecker. Kronecker's initial idea, introduced it in his class on the concept of number (1887), was to give an arithmetic form to all pure mathematics. Klein's standpoint on the movement of arithmetization is that it is a rigorization process arising in analysis after all the discoveries of the 18th Century. Klein explicitly dismisses the "mere putting of the argument into arithmetical form" as unimportant, and considers that only the rigid logic behind the argument is significant. By doing so, Klein reduces an approach heavy with epistemological requisites to a rather narrow demand of rigid logic to secure the argument. But by doing so, he also seems to enlarge the arithmetization approach to any mathematician who demands a more rigorous definition of the fundamental notions of analysis. Historians of mathematics followed Klein's lead and came to consider arithmetization as one large movement of rigorization of the foundations of analysis. Indeed, while it is acknowledged that arithmetization involves many different practices, it is essentially considered to be a large rigorization process in which any non-geometrical characterization of the foundations of analysis can be included. The phrase "arithmetization" seems thus to have lost the 'putting into arithmetic form' part of the approach, which did constitute the core of the idea when it was first introduced by Kronecker in 1887.

I would like to suggest that by dismissing the part of arithmetization that actually gives an arithmetical form to what is studied, we may be overlooking some fairly important components of "arithmetization", and in particular a finely tuned understanding of the conception of arithmetic sustaining arithmetization and its relations with the rigor pursued by arithmetizers. My point, here, will be to underline how arithmetization is a multi-folded movement in which mathematicians pursuing similar goals deploy very different strategies.

In order to give a more precise idea of my argument, I will propose to analyse and explain Richard Dedekind's strategy of arithmetization of mathematics. Dedekind's 1872 construction of the real numbers by means of cuts, in "Stetigkeit und irrationale Zahlen", is usually regarded as an archetype of arithmetization. However, I will suggest that his approach is more generally to be considered as an arithmetization of mathematics. Dedekind's mathematics is indeed guided by the desire to provide certain theories with a rigorous and general definition for the fundamental definitions. I will try to show how generality and rigor, for Dedekind, appear to be reachable the most efficiently through arithmetic, making his approach an arithmetization in the most literal sense of the term. I will proceed to unfold the characteristic elements of Dedekind's strategy of arithmetization: to use the "simplest principles of arithmetic". This approach is first introduced in Dedekind's 1871 theory of algebraic integers and largely developed in the subsequent versions of the theory. I will argue that Dedekind develops a strategy of arithmetization that promotes an abstract arithmetic and does not aim to reduce mathematics to natural numbers. On the contrary, Dedekind encourages the introduction of new concepts and further developments of mathematics, and I will show how the operations of rational arithmetic are used to invent new concepts considered as suited to founding mathematical theories.

To flesh out my argument, I will illustrate the previous statements with elements of the paper on algebraic functions of one complex variable, co-written with Heinrich Weber and published in 1882: "Theorie der algebraischen Funktionen einer Veränderlichen", in which Dedekind and Weber transfer to algebraic functions the arithmetical methodology previously developed in algebraic integers. Their aim is to provide a general and rigorous ground to Riemann's theory of algebraic functions and, indeed, they are able to give a new general definition of the Riemann surface and of the basic notions of the theory by means of solely the "simplest principles of arithmetic."

12:35–1:10pm Andrew Aberdein, Department of Humanities and Communication, Florida Institute of Technology and Matthew Inglis, Mathematics Education Centre, Loughborough University: "Explanation and Explication In Mathematical Practice"

There are a diverse range of accounts of mathematical explanation—the best-known include those of Steiner, Kitcher and van Fraassen [6, 10, 11]. However, each of these accounts has serious problems [3, 4, 8, 9]. This paper proposes a new approach and offers empirical data suggesting that it closely corresponds with mathematical practice.

There is something reflexive to the project of writing about explanation, since ultimately one seeks to explain it. One strategy for diminishing this appearance of circularity originates with the work of Rudolf Carnap. He distinguishes between explanation and explication. Explanation is a relationship between a fact or natural phenomenon and its ostensible explanation. Explication is a process of clarification or conceptual analysis. Carnap proposes four requirements which good explications should meet: 'The explicatum is to be similar to the explicandum...[characterized] in an exact form, ...a fruitful concept, ...[and] as simple as possible' [1]. So the task of the philosopher studying explanation is not to explain it but to explicate it: to construct a simple, fruitful and exact model which is demonstrably similar to the actual practice of explanation. Each account of mathematical explanation claims to have such a model, but so far none of them is a good fit with all four criteria. Perhaps the mistake has been to look too far afield; could explication itself act as mathematical explanation?

As yet no philosopher of mathematics has stated that mathematical explanation just is explication, but some have argued that mathematics is explicatory [2, 7]. Conversely, others have observed that philosophical explication resembles mathematical reasoning [5]. So an explicative account of mathematical explanation is continuous with a plausible position in the philosophy of mathematics.

This paper will discuss new empirical research which provides stronger support for this account. 255 mathematicians were asked to characterise a proof of their choice using 80 different adjectives that have often been used to describe mathematical proofs. A five-point Likert scale was provided for each adjective and the responses were subjected to a principal components analysis. The adjectives lined up on four major dimensions, which we characterized as aesthetics, intricacy, precision, and utility. The study suggests that explanatoriness is a multi-dimensional concept: 'explanatory' was positively correlated with precision and (more weakly) utility, negatively correlated with intricacy, and not correlated with aesthetics. That's to say, the mathematicians who completed the study expected explanatory proofs to be precise, useful and not intricate. Or, in other words, simple, fruitful and exact. The mathematical explanation to be found in mathematicians' descriptions of proofs is, in Carnap's sense, not explanation at all: it's explication.

References

[1] Rudolf Carnap. *Logical Foundations of Probability*. Routledge & Kegan Paul, London, 1950.

[2] Antony Eagle. Mathematics and conceptual analysis. *Synthese*, 161:67–88, 2008.

[3] Johannes Hafner and Paolo Mancosu. The varieties of mathematical explanation. In Paolo Mancosu, Klaus Frovin Jørgensen, and Stig Andur Pedersen, editors, *Visualization, Explanation and Reasoning Styles in Mathematics*, pages 215–250. Springer, Dordrecht, 2005.

[4] Johannes Hafner and Paolo Mancosu. Beyond unification. In Paolo Mancosu, editor, The Philosophy of Mathematical Practice, pages 151–178. Oxford University Press, Oxford, 2008.

[5] Alexander S. Harper. An oblique epistemic defence of conceptual analysis. *Metaphilosophy*, 43(3):235–256, 2012.

[6] Philip Kitcher. Explanatory unification. *Philosophy of Science*, 48:507–531, 1981.

[7] Theo A.F. Kuipers. Mathematics and explication: Reply to Jean Paul Van Bendegem. In Roberto Festa, Atocha Aliseda, and Jeanne Peijnenburg, editors, *Cognitive Structures in Scientific Inquiry: Essays in Debate with Theo Kuipers*, volume 2, pages 170–173. Rodopi, Amsterdam, 2005.

[8] Michael Resnik and D. Kushner. Explanation, independence, and realism in mathematics. *British Journal for the Philosophy of Science*, 38:141–158, 1987.

[9] David Sandborg. Mathematical explanation and the theory of why-questions. *British Journal for the Philosophy of Science*, 49:603–624, 1998.

[10] Mark Steiner. Mathematical explanation. *Philosophical Studies*, 34:135–151, 1978.

[11] Bas Van Fraassen. *The Scientific Image*. Clarendon, Oxford, 1980.

2:30–3:05pm John Baldwin, Department of Mathematics, Statistics and Computer Science, University of Illinois at Chicago: "From Geometry to Algebra and Analysis"

The distinction between counting and measuring dates to antiquity. Since the late 19th century, the foundations of mathematics have generally been taken to proceed from counting to measuring: from the arithmetic of the natural numbers to constructing the real numbers and then both complex numbers and real geometry. We explore the foundations from a different standpoint.

In this talk we describe the first part of the following program.

(1) Establish the theory of real closed (ordered) fields on a geometric basis giving a 'geometric basis' for the theory of similar triangles, proportionality, and area.

(2) Use o-minimality and the addition of suitable functions to found certain parts of analysis.

(3) Develop a similar theory for projective and affine complex geometry.

We build on Detlefsen's notion [3] of 'descriptive completeness' to evaluate the program, not by categoricity, but whether it accounts for a specific fundamental result: the 'side-splitter theorem': a line parallel to the base of a triangle cuts the sides proportionally.

It is well known that any Pappian projective plane π can be coordinatized by a field. More precisely, there is an interpretation of a field F into π whose points form a line in π and so that π is definably a set of pairs from that interpreted field. Now one can recover the first order theory of $\Re$ or $\mathbb{C}$ by imposing on the (definable in π) field operations the requirements that F is real closed field or an algebraically closed field of characteristic 0. We can then recover the affine plane by deleting the line at infinity. This yields the metric structure in the real closed but not the algebraically closed case.

For this talk, we will restrict to a 'Euclidean' axiomatization of plane geometry which leads by a construction of Hartshorne [5] to the coordinatization of the geometry by an ordered field in such a way that the side-splitter theorem holds under the defined multiplication. A short argument for this appears in [1]. Our subtitle refers to the fact that similarity is fundamental to this notion of multiplication (appropriate for fields) which is a very different notion than the inductive definition of multiplication on the natural numbers. In particular, it allows for decidability. Hartshorne's analysis eliminates the famous 'detour' through area in establishing the properties of similarity.

The extensions to items 2 and 3 are deferred to a later talk. Note that by the categoricity of ACF_0, any model of ACF_0 with cardinality $2^{\aleph_0}$ is isomorphic to $\mathbb{C}$ and so admits a topological structure isomorphic to $\mathbb{C}$. Thus in a sense we have a discrete axiomatization of the continuous structure. There is no $2^{\aleph_0}$-categorical axiomatization of $\Re$ so some extension is necessary. However, it may be that something weaker than a categorical axiomatization will suffice. Tarski [4] suggests a 'continuity schema' analogous to that for first-order Peano. Peterzil and Starchenko approach the complex case through o-minimality of the real part [6]. Finally, D'Aquino, Knight, and Starchenko [2] connect nonstandard models of first-order Peano with appropriate models of RCF.

References

[1] John T. Baldwin and Andreas Mueller. A short geometry. `http://homepages.math.uic.edu/~jbaldwin/CTTIgeometry/euclidgeonov21.pdf`, 2012.

[2] P. D'Aquino, J. Knight, and S. Starchenko. Real closed fields and models of arithmetic. *Journal of Symbolic Logic*, 77:1–11, 2012.

[3] M. Detlefsen. Completeness and the ends of axiomatization. In J. Kennedy, editor, *Interpreting Goedel*. Cambridge University Press, to appear.

[4] S. Givant and A. Tarski. Tarski's system of geometry. *Bulletin of Symbolic Logic*, 5:175–214, 1999.

[5] Robin Hartshorne. Geometry: *Euclid and Beyond*. Springer-Verlag, 2000.

[6] Ya'acov Peterzil and Sergey Starchenko. Tame complex analysis and o-minimality. In *Proceedings of the International Congress of Mathematicians*, Vol. II (New Delhi, 2010), page 5881, 2010.

2:30–3:05pm Ashton Sperry-Taylor, Department of Philosophy, University of Missouri: "Modeling The State Of Nature"

Social contract theory investigates the origins of the state and its legitimate authority over individuals. Traditionally, the social contract consists of the individuals' political or moral obligations towards one another. Acting according to said obligations presumably betters individuals' life outcomes. Social contract theorists investigate the nature of the social contract by investigating those obligations individuals will agree to in the state of nature. Those agreements will be included in the contract. Forming agreements between rational individuals is difficult, however. The problem is that it is rational to follow one's self interest in the state of nature against a group's interests, even when the group's interests benefit everyone. This is the *problem of forming the social contract*.

Evolutionary game theorists (such as Brian Skyrms) interpret the aforementioned problem as having individuals choose between two equilibria. One equilibrium conceptualizes one's self interests in the state of nature; the other equilibrium conceptualizes the group's interests.

Evolutionary game theorists attempt to solve this problem in two steps. First, they apply a strategic interaction (games such as the Prisoner's Dilemma or Stag Hunt) that best models the agents' interactions in the state of nature. Second, they apply a dynamic process (such as the replicator dynamics) to the possible interactions, letting programmed agents interact with each other for a period of time, and study the results. The goal is to have an accurate enough model of the state of nature and of the agents' behavior to plausibly explain the emergence of the social contract. The result is that the social contract provides greater evolutionary fitness for individuals (or perhaps the group itself) than the state of nature.

The literature has primarily focused on the second step. This paper, however, investigates the first step. I argue that game theorists have not properly modeled the state of nature. It is not the problem of *choosing* between two equilibria. Instead, agents already *behave* according to the state of nature equilibrium, which is difficult to escape. The problem becomes a study on how agents *escape* the state of nature equilibrium for the social contract equilibrium. Moreover, the nature of the strategic interaction provides a different dynamic process, called stochastic stability. The result is a more pessimistic conclusion on the feasibility of evolutionary game theory solving the problem of forming the social contract.

3:05–3:40pm Sylvain Cabanacq, Department of History and Philosophy of Science, Université Paris Diderot - Paris 7: "Contrasting proofs: categorical tools, model-theoretical methods and the Manin-Mumford Conjecture"

In the mid sixties, Yuri Manin and David Mumford, independently, raised a crucial question of diophantine geometry, concerning the intersection of a curve with the torsion subgroup of its Jacobian. According to Serge Lang [Lan91], Manin was led to this problem by his work on an analogue of the Mordell conjecture, in the case of a function field in characteristic 0. This question, now called the "Manin-Mumford conjecture", can be formulated:

Conjecture 1. Let A be an abelian variety defined over $\mathbb{Q}$ and X a closed subvariety of A. Denote by Tor(A) the set of torsion points of A. Then

$$X \cap Tor(A) = \bigcup_{i \in I} X_i \cap Tor(A),$$

where I is a finite set and each X_i is the translate by an element of A of an abelian subvariety of A, immersed in X.

Several proofs of this conjecture, and generalizations of the proposition, have been given, using different methods and conceptual frameworks, among which:

(1) the first one, in 1983, by Raynaud, in the categorical context of the geometry of schemes [Ray83a];

(2) a proof by Coleman in 1985, using p-adic integration [Col85];

(3) a model-theoretic proof, in 2001, by Hrushovski [Hru01];

(4) a proof in "classical" algebraic geometry, proposed by Rössler and Pink in 2002 [PR02].

In this chapter, we would like to present, in more details, two of these proofs: the scheme-theoretic one by Raynaud and the model-theoretic one by Hrushovski, in order to contrast the different tools and concepts used. Indeed, the two proofs use apparently independent methods, from category theory, "à la Grothendieck", for the former and from the model theory of fields for the latter. But is it possible to "translate", term-by-term, one proof into the other?

We can see that the authors claim two different kinds of "generality", that of the schematic language on the one hand and on the other hand that of a logical context that allows the model-theoretical proof, according to Hrushovski, not to "deal first with the points of order relatively prime to a given p", as it's the case for the others. So we would like to analyse the practical strategies of proof underlying these claims, in order to clarify the different meanings of "generality" involved.

Moreover, it is important to notice that the context of diophantine geometry induces, in the two proofs, two different descriptions of what can be seen as "geometrical", in terms of sheaves or in terms of the abstract model-theoretical concepts of dimension, independence and local modularity. So, can we define precisely the connections between these two ideas of "geometry"? This question allows us to see in a concrete case the debate between sheaves and model-theory raised by McIntyre in [McI03]:

> none of the main texts [in model theory] uses in any nontrivial way the language of category theory, far less sheaf theory or topos theory. Given that the most notable interaction of model theory with geometry are in areas of geometry where the language of sheaves is almost indispensable (to the geometers), this is a curious situation.

A first step towards such connections could be the "second-order arguments" permitted, according to Hrushovski, by model theory, concerning "properties of the class of all the definable sets".

[Bou98] Elisabeth Bouscaren, editor. *Model Theory and Algebraic Geometry, An introduction to E. Hrushovski's proof of the geometric Mordell-Lang conjecture*, volume 1696 of Lecture Notes in Mathematics. Springer, 1998.

[Bou00] Elisabeth Bouscaren. Théorie des modèles et conjecture de Manin-Mumford (d'après Ehud Hrushovski). In *Séminaire Bourbaki*, volume 1999-2000. mars 2000.

[Col85] Robert Coleman. Torsion points on curves and p-adic Abelian integrals. *Annals of math.*, 121 :111–168, 1985.

[Hru01] Ehud Hrushovski. The Manin-Mumford conjecture and the model theory of difference fields. *Ann. Pure. App. Logic*, 112(1) :43–115, 2001.

[Lan91] Serge Lang. *Number theory III : Diophantine geometry*, volume 60 of *Encyclop. Math. Sci.* Springer-Verlag, 1991.

[McI03] Angus McIntyre. Model theory: geometrical and set-theoretic aspects and prospects. *The Bulletin of symbolic logic*, 9(2) :197–212, 2003.

[PR04] R. Pink and D. Rössler. On psi-invariant subvarieties of semiabelian varieties and the Manin-Mumford conjecture. *J. Algebraic Geom.*, 13(4) :771–798, 2004.

[PZ03] A. Pillay and M. Ziegler. Jet spaces of varieties over differential and difference fields. *Selecta Math.*, 9(4) :579–599, 2003.

[Ray83a] Michel Raynaud. Courbes sur une variété abélienne et points de torsion. *Invent. Math.*, 71(1) :207–233, 1983.

[Ray83b] Michel Raynaud. Sous-variétés d'une variété abélienne et points de torsion. In *Arithmetic et geometry*, volume I, pages 327–352, Boston, 1983. Birkaüser.

[RP02] D. Rössler and R. Pink. On Hrushovski's proof of the Manin-Mumford conjecture. In *Proceedings of the International Congress of Mathematicians (Beijing, 2002)*, volume I, pages 539–546, Beijing, 2002. Higher Ed. Press.

[Zil10] Boris Zilber. *Zariski Geometries: Geometry from the Logician's Point of View*, volume 360 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, 2010.

3:05–3:40pm Luca San Mauro, Center of Philosophy, Scuola Normale Superiore, Pisa: "Algorithms, formalization and exactness: a philosophical study of the practical use of Church- Turing Thesis"

Even though its theoretical status has been widely investigated in philosophy (see, for example, [2]), there is very little literature regarding the practical use of Church-Turing Thesis (CTT). This could be surprising. An appeal to CTT (somehow implicit) can already be found in a celebrated paper by Post in 1944 as a preliminary justification for the unexpected flavour of informality in which his proofs are given (see [3]). In 1967, a standard textbook on classical computability (Rogers, [4]) define "proofs by Church's Thesis" those proofs which rely on informal methods, i.e. proofs that employ informal techniques (generally considered as effective) in the description of an algorithm.

Therefore, the practical use of CTT consists in the gap between the semi-formal language in which algorithms are commonly given in computability theory, and their formal representation in one of the classical models of computation. So it is clear that, even if related, theoretical and practical concerns on CTT are not coincident. Typical questions on the theoretical side may regard the ontological status of CTT, or its correct placement in the mathematical discourse (e.g., if it can be object of some formal proof); on the other side, practical aspects focus on the way in which mathematical practice has been shaped by the use of CTT. Our aim is to face this latter class of problems. In doing so, we proceed as follows.

The first part of our work is devoted to an historical overview of those practical aspects. We show how the employment of CTT—with its consequent admission of a certain amount of informality—gave rise to an assorted class of proof techniques that led in turn to the formulation of the peculiar degree of formalization of computability theory. On this concern, classical methods (such as priority and finite injury) are brought into consideration, and the very notion of method is also discussed.

In the second part of the work, we rephrase some of these ideas in a more general context. Such blend of formal and informal aspects within the language of computability theory is of course analogue to the case of every other main theory in contemporary mathematics—nonetheless, taking computability as a specific case-study is interesting because it allows to focus on the birth and the quick development of a mathematical language with its embedded rules. With this perspective in mind, we claim that the presence of informality, in mathematical practice, is to some extent unavoidable and that every theory has, among its most fundamental tasks, that of fixing its own level of formalization. Thus, in conclusion, we believe that any attempt to give a general model of real mathematical proofs (see [1]) can be fruitful only by taking account of this dynamic scenario.

References

[1] Leitgeb, H. (2009) On informal and formal provability in *New waves in the philosophy of mathematics* (Bueno, O. and Linnebo, O. ed.), Palgrave Macmillan.

[2] Olszewski, A. (ed.) (2007) *Church's Thesis after 70 Years*, Ontos Verlag.

[3] Post, E. (1944), Recursively enumerable sets of positive integers and their decision problems, *Bulletin of American Mathematical Society*, 50, 284-316.

[4] Rogers, H. (1967), *Theory of recursive functions and effective computability*, McGraw-Hill.

4:15–5:30pm Chris Pincock, Department of Philosophy, The Ohio State University: "Felix Klein as a Prototype for the Philosophy of Mathematical Practice"

In this paper I aim to determine the contemporary relevance of the conception of mathematics championed by Felix Klein (1849-1925). Klein is best known among philosophers for his "Erlanger Program" in geometry. However, Klein also devoted a great deal of energy to articulating and defending a philosophical approach to mathematics that went far beyond geometry. Among other things, Kline emphasized the illuminating connections between different areas of mathematics, e.g. the *Lectures on the icosahedron and the solution of equations of the fifth degree* (1884). A central component of this approach is the interdependence of intuition and logic in the development and clarification of mathematical results. I hope to determine what exactly Klein thought the role of intuition was supposed to be. Is it merely of psychological or pedagogical importance, or does intuition play a role in justifying new mathematical claims? I will also consider the more radical possibility that intuition provides essential insight into the nature of the subject-matter of mathematics itself. In the end, Klein may offer an instructive prototype for more recent attempts to defend the philosophical significance of non-foundational mathematics.

October 5

9:00–10:15am Colin McLarty, Department of Philosophy, Case Western Reserve University: "Proofs in practice"

The talk describes the practical impact of three proofs. By this I mean the proofs and not the theorems proved. First is Euclid's proof of the Pythagorean theorem in contrast to the likely earlier proof by Hippocrates of Chios. Second is Hilbert's proof of the finite generation of invariants which Gordan called "not mathematics but theology." Third is the yet unknown proof in Peano Arithmetic of Fermat's Last Theorem.

10:15–10:50am Ken Manders, Department of Philosophy, University of Pittsburgh: "Problems and Prospects for Philosophy of Mathematical Understanding"

Mathematics is first and foremost a means of understanding. Some work is required to cast "means of understanding" as a proper philosophical topic, and perhaps many have been put off by this. There is, however, little difficulty in dealing with this in at least a preliminary way, that should entitle investigation to proceed.

Case studies then allow one to gather insights as to what promotes understanding in mathematics. Start with contrast situations in which something is poorly respectively better understood. One need encounter no philosophically significant (rather than just mathematical) difference, but one does.

What emerges is a disparate (desperate?) variety of phenomena: many different kinds of things seem to contribute to an in some respect improved mathematical understanding. A few have long been understood, e.g., logical systematization. Most of them involve some kind of conceptual recasting, and we may be little used to theorizing on relevant aspects of conceptual recasting, but may learn if we try.

Prospects, however, for mathematical understanding being "one thing", that philosophers might define, are so far not good. Nor can we expect that every intelligibility difference has a philosophically interesting explanation.

Among the ideas that emerged from our work: differences in the character of expressive means (even with complete inter-translatability in principle) can be decisive, for example, to what uniformities of reasoning are attainable. Moreover, at different stages of an argument different expressive means may best promote intelligibility.

A direction less investigated is how intelligibility is often effected by putting a claim in a theoretical context.

Some relevant outcomes can be well treated by logic (axioms, proof, computability); but not all need be, and in any case the nature of the transition to the more favorable state tends to lie beyond the scope of such ideas. These investigations are also more difficult because examination of single proofs may completely fail to bring out the nature of the transformed situation. This is analogous to the difficulties in giving an account of "theoretical" (vs individual-fact) explanation in the Philosophy of Science: we need to characterize features of global conceptual structure rather than of individual arguments.

10:15–10:50am Ramzi Kebaili, Department of History and Philosophy of Science, Université Paris Diderot - Paris 7: "Examples of 'synthetic a priori' statements in mathematical practice"

Kant was the first philosopher to establish the distinction between "analytic" and "synthetic" statements. According to him, mathematics provides a good example of "synthetic a priori" statements. But what did he exactly mean by that? By reviewing his own examples, I will argue that he had in mind (at least) two different features of mathematical practice that cannot be correctly understood by a strictly logical view of proof. Following Kant's inspiration, other philosophers such as Poincaré or Wittgenstein argued that mathematical proofs are "synthetic a priori" and need some kind of "intuition" to be carried out. During the talk we will closely examine five distinct examples raised by these three philosophers : "5+7=12", "There is no biangle", the Principle of Induction, the Invariance of Dimension Theorem, and the Prime Number Theorem. I will argue that these examples give five distinct reasons why mathematics can not be said to be analytic, and so five different meanings for the notion of "synthetic a priori".

11:20–11:55am Susan Vineberg, Department of Philosophy, Wayne State University: "Are There Objective Facts of Mathematical Depth?"

Maddy argues that there are two interpretations of mathematics that are compatible with the version of naturalism (Second Philosophy) that she endorses. One of these is a minimal version of realism (Thin Realism) according to which mathematics is a body of truths about mind-independent objects, while the other is a non-realist view (Arealism) that rejects these claims. On both interpretations, mathematics is an objective practice with objective standards of proof and axiom selection. In particular, Maddy claims that there are objective facts of mathematical depth governing axiom choice and the development of mathematical theory. For the Thin Realist these facts undergird the truths of mathematics, whereas for the Arealist they are merely facts that constrain practice. From the perspective of Second Philosophy, it is not the differences between Thin Realism and Arealism concerning the existence of mathematical objects and mathematical truth that are important, but the shared claim that there are objective facts of mathematical depth. Despite their central role in her account of mathematical practice, Maddy says very little about what the facts of mathematical depth amount to. This paper explores the question of how mathematical depth is to be understood. A variety of distinct mathematical goals that can be associated with mathematical depth are elaborated, which in turn raises questions for the objectivist about the trade-offs between them. Finally, I argue that Second Philosophy lacks the resources to respond to this problem, and more generally to defend the claim that there are indeed such facts of mathematical depth.

11:20–11:55am Oran Magal, Department of Philosophy, McGill University: "Teratological Investigations: What's in a monster?"

Hermite complained about such 'monsters' as functions everywhere continuous and nowhere differentiable. Poincaré quipped that "logic breeds monsters". Taking up this terminology, Lakatos devoted parts of his Proofs and Refutations to discussing the notion of 'monster-barring' definitions. The question I would like to discuss in this talk is, what makes the 'monsters' monstrous?

Arguably, this is not a matter of subjective taste or psychology, but rather, that this has to do with questions of legitimation and disciplinary standards, which come to the fore especially strongly during what J. Gray (and others) have called the modernist transformation of mathematics; briefly, the increasing independence of pure mathematics from potential applications, and the discussion of mathematical questions not necessarily motivated by physical problems and the like, bring up questions about legitimacy and coherence. This topic rewards not only an historical but also a philosophical analysis of this discourse of justification.

As an example, aside from obvious choices such as the introduction of Cantorian transfinite cardinalities, or indeed the 'monsters' castigated by Hermite, one can consider the Banach-Tarski Paradox, and the sense in which it is a paradox at all, as opposed to that in which it is part of a broader transformation or expansion of the notions of 'area', 'volume', integration and measure. What I argue such examples reveal is the involvement of a kind of cultivated intuition, trained judgement internal to a discipline, and which is not always universally agreed on. This idea, if accepted, could have interesting epistemological uses.

11:55–12:30pm Bernd Buldt, Department of Philosophy, Indiana University-Purdue University Fort Wayne: "Mathematics—Some "Practical" Insights"

Mathematicians in particular have demanded that philosophers of mathematics should pay more attention to what mathematics is and to what practitioners in the field are doing. Familiar names in this context are Gian- Carlo Rota, who was a strong advocate of a Husserlian approach,[1] and Frank Quinn who, along with Arthur Jaffe, helped

[1] See, e.g., [Rota et al., 1997] pt. II and [Rota, 2008], pt. II.

to spark what was later called the "Jaffe-Quinn Debate".[2] Rota, however, never progressed beyond programmatic announcements[3], while Quinn (due, probably, to his research topics) continued to exhibit a strong affiliation with some sort of an axiomatic foundationalism.[4]

Some initial progress was made by the predominantly European PhiMSAMP network[5], but concrete mathematical case studies are still few and far between[6], while some tried to give it a general philosophical backing.[7]

We propose a different approach, which we take to be complementary to the strands of work just mentioned. First, we conceive of mathematics as a trade that is defined by common practices; in distinction to PhiMSAMP, we frame these practices in terms of technical practices (e. g, proof techniques) and couch them to a lesser degree in sociological terms. Second, we neither assume nor expect uniformity across fields of mathematics. Rather, we start with the *Mathematics Subject Classification* (MSC)[8], compiled and agreed upon by the editorial offices of both *Mathematical Reviews* and *Zentralblatt MATH* (formerly, *Zentralblatt für Mathematik und ihre Grenzgebiete*) and select a subset of the 97 areas the MSC lists. This subset is aimed at diversity but also at a balanced representation of what are considered the main branches of mathematics (e.g., discrete math, algebra, analysis, geometry & topology, applied math). We then extract from recent textbooks in this subset practices that seem apt to characterize that particular field.

This abstract describes work in progress.

References

[Carter, 2005] Carter, Jessica: "Individuation of objects a problem for structuralism?," in *Synthese* vol. 143, No. 3, pp. 291–307 (2005).

[Carter, 2008] Carter, Jessica: "Structuralism as a philosophy of mathematical practice," in *Synthese* vol. 163, No. 2, pp. 119–131 (2008).

[Kerkhove/Bendegem, 2007] Van Kerkhove, Bert & van Bendegem, Jean Paul (eds): *Philosophical Perspectives on Mathematical Practice: Bringing Together Philosophy of Mathematics, Sociology of Mathematics, and Mathematics Education*, Dordrecht: Springer (2007).

[Kerkhove, 2009] Van Kerkhove, Bert (ed): *New Perspectives on Mathematical Practices: Essays in Philosophy and History of Mathematics*, Singapore: World Scientifc Publishing (2009)

[Löwe/Müller, 2010] Löwe, Benedikt & Müller, Thomas (eds): *PhiMSAMP. Philosophy of Mathematics: Sociological aspects and mathematical practice*, London: College Publications (2010).

[Maddy, 2007] Maddy, Penelope: *Second Philosophy: A Naturalistic Method*, New York: Oxford (2007).

[2]See [Quinn, 1993].
[3]Cf., however, [Palombi, 2003].
[4]See [Quinn, 2012], [Quinn, A].
[5]See [Kerkhove/Bendegem, 2007], [Kerkhove, 2009], and [Löwe/Müller, 2010].
[6]See, e.g., [Carter, 2005] or [Carter, 2008].
[7]See [Maddy, 2007], pt. IV.
[8]See [Wegner/Werner 2010] or `http://msc2010.org/`.

[Palombi, 2003] Palombi, Fabrizio: *The Star and the Whole: Gian-Carlo Rota on Mathematics and Phenomenology*, Boca Raton, FL: CRC Press (2011); transl. from the Italian, *La stella al'intera: la ricerca de Gian-Rota tra matematica e fenomenologia*, Torino: Bollati Boringhieri (2003).

[Quinn, 1993] Quinn, Frank & Jaffe, Arthur: " 'Theoretical mathematics': Toward a cultural synthesis of mathematics and theoretical physics," in *Bull. Am. Math. Soc.*, NS, vol. 29, No. 1, pp. 1–13.

[Quinn, 2012] Quinn, Frank: "A revolution in mathematics? What really happened a century ago and why it matters today," in *Notices Am. Math. Soc.* vol. 59, No. 1, pp. 31–37.

[Quinn, A] Quinn, Frank: *Contributions to a Science of Contemporary Mathematics*, unpublished manuscript available at: `www.math.vt.edu/people/quinn/history nature/nature0.pdf`.

[Rota et al., 1997] Rota, Gian-Carlo & Kac, Marc & Schwartz, Jacob T.: *Discrete Thoughts*, Boston: Birkhäuser (1993).

[Rota, 2008] Rota, Gian-Carlo: *Indiscrete Thoughts*, ed. by Fabrizio Palombi, Boston: Birkhäuser (2008).

[Wegner/Werner 2010] Wegner, Bernd & Werner, Dirk (eds): *Mathematics Subject Classification 2010*, ed. European Mathematical Society, Fachinformationszentrum Karlsruhe, Heidelberger Akademie der Wissenschaften, Berlin: Springer (2010); available at `http://msc2010.org`.

11:55–12:30pm Madeline Muntersbjorn, Department of Philosophy, University of Toledo: "Cognitive Diversity and Mathematical Progress"

Jeremy Gray (2012) writes that Poincaré's 1898 essay "On the foundations of geometry" "can be read as an early example of cognitive science" (49). Poincaré suggests that our preference for Euclidean geometry originates in our constitution as an evolved kind. He argues not only that our experiences as rigid bodies in the world incline us towards one kind of geometry, but also that if our experiences had been different, we would have different inclinations. Further, our minds are not all cast in the same mold, to use an old metaphor for cognitive diversity. In his last public lecture in 1912, Henri Poincaré spoke at the inaugural meeting of *Le Ligue Française d'Éducation Morale*. In his talk, "L'Union Morale," he said, "Let us guard against imposing uniform methods on all; that is unrealizable and, moreover, it is not desirable. Uniformity is death because it is a door closed to all progress..."[9] One might read this remark as a rhetorical olive branch, offered by a scientist to an audience of wary humanists. Yet, Poincaré's commitment to intellectual diversity is central to his view of mathematics as something alive that grows. This paper considers Poincaré's conjecture that mathematicians come in different kinds. Poincaré distinguishes between logicians, who correct and prove, and intuitionists, who hypothesize and predict: "The two sorts of minds are equally necessary for the progress of science; both the logicians and the intuitionalists have achieved great things that others could not have done."[10] Contemporaries of Poincaré

[9] *Last Essays*, p. 116. "Gardons-nous d'imposer à tous des moyens uniformes, cela est irréalisable, et d'ailleurs, cela n'est pas à désirer: l'uniformité, c'est la mort, parce que c'est la porte close à tout progress" (*Dernières pensées*, p. 256).

[10] *The Value of Science*, p. 17. "Les deux sortes d'esprits sont également nécessaires aux progrès de la science; les logiciens, comme les intuitifs, ont fait de grandes choses qeu les autres n'auraient pas pu faire" (*La valeur de la science*, p. 10).

also argued that different people had different kind of minds. But while Pierre Duhem contrasts Blaise Pascal's *espirit de finesse* and *esprit de géométrie* as cultural kinds, Poincaré contrasts *intuitifs* and *analysts* as individual kinds of mathematical minds that may be born into any culture. Significantly, we can trace a path from these rudimentary speculations to contemporary cognitive neuroscience investigations into the genetic bases for visual reasoning, e.g., Delis hierarchical processing tasks. This talk will conclude by considering why, according to Poincaré, cognitive diversity is necessary for the growth of mathematical knowledge.

2:00–2:35pm Michele Friend, Department of Philosophy, George Washington University: "Using a Paraconsistent Formal Theory of Logic Metaphorically"

I make two points. First, pluralists in the philosophy of mathematics compare together inconsistent mathematical theories and philosophies of mathematics without deciding in advance that one is better or correct. In some of the discussions they also use both inconsistent theories together. So, the 'use-mention' distinction is not always upheld. Therefore, the threat of triviality is immediate. This calls for their use of a paraconsistent logic to guide their reasoning.

Second, the use made of paraconsistent logic is usually metaphorical, and not direct. One example of the indirect use of a logic is a dialectical use. I shall characterise the dialectical use and give an example.

2:00–2:35pm José Ferreirós, Department of Philosophy and Logic, University of Seville and Elías Fuentes Guillén, Department of Philosophy, Logic and Aesthetics, University of Salamanca: "Bolzano's 'Rein analytischer Beweis...' reconsidered"

There is an established consensus among historians of mathematics concerning the relevance of Bolzano's 1817 paper 'Rein analytischer Beweis...' for the development and foundations of modern analysis. Renowned for the way it advanced in the direction of the 'modern' rigorous analysis of Cauchy and Weierstrass, Bolzano concretely is credited for having provided there:

(1) a correct definition of the continuity of a one real variable function (Preface);

(2) an adequate criterion for the pointwise convergence of an infinite sequence, although an insufficient proof thereof (§7);

(3) an initial formulation of the Bolzano-Weierstrass theorem (§12);

(4) an almost complete proof of the intermediate value theorem (§15).

In fact, concerning the fourth point, it has always been stressed that while the proof's idea was a good one, Bolzano's argument, "prior to any definition or construction of the real numbers, is inevitably inadequate" (Russ 1980, 157).

Nevertheless, a careful examination of that text throws some doubt on the proper assessment to be made, both due to Bolzano's understanding of his program by that time and because of the quantity core idea's character he is dealing with. It is normally pointed out that the gap in Bolzano's arises from lacking a precise notion of a real number, evaluation which seems to us anachronistic, assuming as it does that his proof's conception was in the line with, say, the Weierstrassian approach.

Apparently not enough effort has been devoted to clarify the structure of Bolzano's argument as well as the assumptions and goals of his practice in the field of analysis. His program was partly inherited from common views on mathematics and analysis around 1800, but at the same time it had groundbreaking features. In our view, the faithful reading of Bolzano (1817) must be as a transitional work not yet conceived in what we call the static perspective of 'arithmetized' analysis (Weierstrass, Dedekind), since it still features important traits of the dynamic conception in Newtonian tradition. Moreover, he was a key figure in the transition from that older dynamic conception of quantities to the modern static conception of number.

We shall focus our attention on the lacuna in §7 of (Bolzano 1817) and try to provide an adequate interpretation of this passage. In particular, we argue that establishing the existence of a real number that satisfies the proposition of the theorem is not actually the aim of Bolzano's argument: rather than focusing on existence, he is discussing the nature of the quantity in question. The ultimate reason for this can be found in the role of the fundamental notion of "variable quantity" in the analysis practice to which Bolzano still belonged at the time, in a sense still opposed to the Weierstrass-Dedekind point of view.

As far as time allows, we will survey the context of Bolzano's aforementioned publication, attending the state of mathematics at Central and Western Europe by the end of the 18th century and the beginning of the 19th; consider Bolzano's influences and sources; and provide a faithful reading of Bolzano's 'Rein analytischer Beweis...', immanent so to speak, leaving aside subsequent work that uses any of the ideas contained in it (even his own). Additionally, we offer some hints of the reinterpretation of his proof in the context of Weierstrassian foundations, which precisely has the effect of dislocating it from its original dynamic setting and reinterpreting within the new, static conception of the real number system.

A history of the development of modern analysis is too well known to bear a repetition of it here.

2:35–3:10pm Jacobo Asse Dayán, Program in Philosophy of Science, Universidad Nacional Autónoma de México: "The Intentionality and Materiality of Mathematical Objects"

I combine the ideas of Julian Cole regarding the intentionality of mathematical objects with those of Madeline Muntersbjorn and Helen de Cruz regarding the constitutional role that external representation plays in the emergence of mathematical objects, and obtain an artifactual theory of mathematical objects. I contend that mathematical objects are epistemic artifacts, as real as everyday artifacts, thus assuaging the mathematical realists' concerns, while evading the epistemological problems of Platonism. In this way, I satisfy the concerns expressed in Benacerraf's famous dilemma, by providing a referential semantics and a naturalistic epistemology for mathematics. What is sacrificed in this account is the perceived necessity and atemporality of mathematical objects, since both the intentionality of the mathematical community and the materiality of mathematical notation are practice dependent phenomena, and thus temporal and contingent.

2:35–3:10pm Philip Ehrlich, Department of Philosophy, Ohio University: "A Re-examination of Zeno's Paradox of Extension"

The real number system was dubbed the *arithmetic continuum* because it was held that this number system is completely adequate for the analytic representation of all types of continuous phenomena. In accordance with this view, the *geometric linear continuum* is assumed to be isomorphic with the arithmetic continuum, the axioms of geometry being so selected to ensure this would be the case. Since its inception, however, there has never been a time at which this now standard *Cantor-Dedekind philosophy of the continuum* has enjoyed the complete allegiance of philosophers or mathematicians. One complaint that was, and to some extent still is, a stumbling block to the universal acceptance of the theory is the contention that the Cantor-Dedekind philosophy of the continuum is committed to the reduction of the continuous to the discrete, a program whose philosophical cogency, and even logical consistency, has been called into question since Zeno contended some 2500 years ago that a geometrical magnitude could neither be constituted from, nor decomposed to, points, on pain of contradiction.

Several attempts have been made over the centuries to resolve Zeno's Paradox of Extension. Of these, the best known and most influential is the one due to Adolf Grünbaum. Grünbaum, in what has come to be the received view, finds the resolution of Zeno's paradox of extension in the tools of real analysis, in the *Lebesgue theory of measure*, in particular. Moreover, to emphasize the central role played by measure theory in Grünbaum's now standard treatment, Brian Skyrms has gone so far as to rename Zeno's paradox of extension, "Zeno's Paradox of Measure."

However, Zeno's Paradox of Extension may be, and we believe should be, construed as a general paradox regarding infinitely divisible geometrical extension, not merely as a paradox of extension in classical real space. But at a minimum this raises serious questions about the generality of Grünbaum's treatment since it appeals to considerations regarding the classical continuum that are not applicable to the more general case. In our talk, we will present a novel and altogether elementary alternative to Grünbaum's famous analysis that does not suffer from this limitation. More specifically, our resolution will not only be applicable to the lines of continuous Euclidean (and non-Euclidean) geometry, but to the lines of models of elementary Euclidean geometry (and non-Euclidean) geometry more generally, as well as to a vast array of other infinitely divisible lines studied by contemporary geometers including those of semi-Euclidean geometry, semi-hyperbolic geometry, semi-elliptic (or non-Legendrean) geometry, and non-Desarguesian geometry to name only a few. Some of these lines, like the classical linear continuum, contain an uncountable infinity points and are devoid of infinitesimal segments, while others contain a countable infinity points and/or infinitesimal segments. We will further argue that our solution, unlike Grünbaum's, is not merely a *formal solution*, but a *philosophical solution* in that our solution not only blocks the paradox by indicating which apparently unexceptional Zenonian premise must be disallowed, but also supplies an explanation of why that premise is, despite appearances, exceptionable. That is, our solution, unlike Grünbaum's, will show that the rejected premise is one to which there are objections independent of its leading to paradox. At the heart of our resolution will not be the theory of Lebesgue measure, a generalization of the theory of length measurement developed for the needs of real analysis, but the theory of length measurement itself. Indeed, we will show that far from establishing the internal inconsistency of the standard theory of geometrical magnitude, the modern day Zenonian arguments are based on an assumption that is incompatible with the standard theory of geometrical magnitude itself and the nature of length measurement in particular. The same will also be shown to be the case for the classical geometry of Euclid.

3:40–4:15pm Rochelle Gutiérrez, Department of Curriculum and Instruction, University of Illinois at Urbana-Champaign, "What is Mathematics? The Roles of Ethnomathematics and Critical Mathematics in (Re)Defining Mathematics for the Field of Education"

Mathematics in schools today still reflects an Enlightenment period. Students are subjects and mathematics is a tool to confirm a given empirical world, a form of authority over the subject (Radford, 2011). Yet, over the past decade, the field of mathematics education has moved beyond a focus on cognitive and social frames to consider more philosophical issues, including morality and politics (Ernest, 2007, Lerman, 2011, Gutierrez, 2013, Restivo, 2007; Valero & Zevenbergen, 2004). This sociopolitical turn, whereby issues of identity and power are now becoming more prominent in theoretical frameworks, opens the door for educational researchers to depict mathematics in potentially new ways. For example, researchers in the field of ethnomathematics have highlighted the fact that mathematics is a human, evolving practice, rather than a static set of theorems and postulates that were discovered long ago, primarily by white, European men (D'Ambrosio, 2006). More than just documenting ancient cultural practices, ethnomathematics takes up current struggles throughout the world whereby indigenous people embody a different view of mathematics, and asks us to reinvision a form of mathematics that is more humane (Knijnik, 2007). Similarly, critical mathematics education researchers have focused largely on issues of power, highlighting the fact that mathematics holds an unearned position of authority and privilege in society; that it formats our lives and implies mathematics is the arbiter of truth (Skovsmose & Yasukawa, 2004); that it creates a false reality and contributes to oppression (Rotman, 1980; Walkerdine, 1994). One offshoot of this work is a move within schools to use mathematics as a tool for modeling and analyzing the injustices in society. However, the fields of ethnomathematicians and critical mathematics often remain separate, often drawing upon different philosophies and epistemologies.

This paper examines in what ways do ethnomathematics and critical mathematics redefine what mathematics is? Are these views consistent? In what ways are ethnomathematics and critical mathematics complicit with the prevalent image of mathematics that rests upon a Western view of rationality and a privileging of abstraction? Implications for education are made.

References

Ernest, P. (2007). The philosophy of mathematics, values, and Keralese mathematics. *Philosophy of Mathematics Education Journal*, vol. 20. Available at `http://people.exeter.ac.uk/PErnest/`

Gutiérrez, R. (2013). The sociopolitical turn in mathematics education. *Journal for Research in Mathematics Education*, 44(1), 37-68.

Knijnik, G. (2007). Mathematics education and the Brazilian Landless Movement: three different mathematics in the context of the struggle for social justice. *Philosophy of Mathematics Education Journal*, 21, 1-18.

Radford, L. (2011). *Education and the illusions of the emancipated self: From subjects to subjectivities*. A keynote address given at the Mathematics Education in Contemporary Society conference. Manchester, England.

Restivo, S. (2007). Theory of mind, social science, and mathematical practice. In B. van Kerkhove & J. P. van Bendegem (Eds.), *Perspectives on mathematical practices: Bringing together philosophy of mathematics, sociology of mathematics, and mathematics education* (pp. 83–106). Dordrecht, the Netherlands: Springer.

Skovsmose, O. and Yasukawa, K. (2004). Formatting power of "mathematics in a package": a challenge for social theorizing? *Philosophy of Mathematics Education Journal*, 15. Available at `http://people.exeter.ac.uk/PErnest/`

Valero, P., & Zevenbergen, R. (2004). *Researching the socio-political dimensions of mathematics education: Issues of power in theory and methodology*. Norwell, MA: Kluwer.

Walkerdine, V. (1994). Reasoning in a post-modern age. In P. Ernest (Ed.), *Mathematics, education and philosophy: An international perspective* (pp. 61–75). Bristol, PA: Falmer.

3:40–4:15pm Danielle Macbeth, Department of Philosophy, Haverford College: "Rigor, Deduction, and Knowledge in the Practice of Mathematics"

The fruits of mathematical inquiry seem to be at once rigorous and ampliative. Kant famously accounted for this, as he would put it, the synthetic a priori character of mathematical judgments, by appeal to what he describes as an intuitive use of reason, the employment of which enables the construction of concepts in pure intuition. Developments in mathematics in nineteenth-century Germany decisively showed that it is instead reason in its discursive use, directly from concepts, that is needed in mathematics, at least in the new form of mathematical practice championed by Riemann. All distinctively mathematical modes of reasoning can be replaced by strictly logical, deductive steps. But how can deductive reasoning extend our knowledge? And if the reasoning is not really deductive, if replacing the distinctively mathematical steps with purely logical ones somehow changes in an essential way the nature of the proof, as Poincaré famously held, then how can mathematical proof be properly rigorous? If the proof is deductive then it cannot be ampliative. If it is not deductive then it cannot be fully rigorous. And yet it *seems* to be *both*, both rigorous and ampliative. The task is to understand how this can be.

4:15–5:30pm Marco Panza, Institute of History and Philosophy of Science and Technology, Université Paris 1 Panthéon-Sorbonne: "On the Epistemic Economy of Formal Definitions"

The aim of the talk will be elucidating the notion of epistemic economy in mathematics, especially in the case of formal definitions of domains of mathematical items, like natural or real numbers.

The basic idea I shall defend is that different such definitions can have different epistemic costs, and, then, some of them, be preferable to other, for minimizing these costs. Broadly speaking, I consider a formal definition to be epistemically more economic than another if its understating calls for less or more basic intellectual resources. In my talks, I'll try to make clear this idea basing on the admission that a formal mathematical theory is aimed to re-cast a certain body of informal mathematics. Two different theories, involving two different definitions of the same items—that is, of items intend to recast the same objects involved in a common body of informal mathematics—can involve a re-casting of different pieces if this body. One can for example, define real numbers by passing through a previous definitions of natural, integers and rational ones, or by only appealing to natural ones, or as ratio of magnitudes, as suggested by Frege. Following these different strategies can result in getting definitions with different epistemic costs. I shall also suggest a classification of formal definitions of domains of mathematical items, according to some of their epistemic features.