Wednesday June 30

Invited Speaker: Dirk Schlimm (McGill University, Canada)
Title: From mathematical content to symbols - and back again
Abstract: It is a common view about mathematical writings, such as symbols, notations, and diagrams, that they are introduced to represent some given mathematical content. For example, abstract numbers are represented by various systems of numerals, the logical structure of propositions is represented by various formal systems of logic, and relations between geometric objects are represented by Euclidean diagrams. However, the notion of a "given mathematical content" is more problematic that it might at first seem. By taking a closer look at episodes from the historical development of mathematics and logic I will discuss how something like a stable mathematical content can emerge through the interactions between intended meanings and symbolic manipulations within and across mathematical practices. More generally, I argue that the relation between content and representational vehicle is better understood as a feedback loop, instead of as a one-way street, as suggested by the common view mentioned above.

Session 1

Speaker: José Antonio Pérez Escobar (ETH Zurich, Switzerland)
Title: Deflating distinctively mathematical explanations in biology: mathematics as enabler of teleological/functional explanations
Abstract: Recently, the issue of whether there are “genuine mathematical explanations” or “distinctively mathematical explanations” outside mathematics has received substantial attention. Lange argued that some explanations are distinctively mathematical because mathematical necessity, and not causes, has most of the explanatory burden (Lange 2013). Craver and Povich have argued that those examples are better explained by causal mechanisms since there is a concrete explanatory directionality for which mathematics, being directionless, cannot account (Craver and Povich 2017). In this work, I address one of these paradigmatic examples, the honeycomb, and propose an alternative interpretation of the role of mathematics more in line with biological practices and scientific intuitions.
This interpretation entails a nominalistic reconstruction of the explanation, which does not require a mathematical object as explanans and hence does not qualify as a “distinctively mathematical explanation”. At the basis of this reconstruction there is a pertinent differentiation between mathematical optimization and biological optimization. The mathematical optimization argument at stake provides a rough “summary” of what is actually a process of biological optimization of a given evolutionary trajectory, with its actual bifurcations resulting from natural selection and the feasible biological solutions involved.
In this context, the mathematical part of the explanation: 1) is a heuristic for a given evolutionary trajectory and depends on specific teleological notions in order to be explanatorily relevant, and 2) assists in the discovery of specific purposes of biological phenomena. I will argue that, in biology, the role of mathematics in those paradigmatic examples of “distinctively mathematical explanations” is actually one of heuristic enablement of teleological/functional explanations.
I will support this alternative interpretation by extrapolating it to a similar but more illustrative case: the hexagonal periodicity of grid cell activity. I will show how similar mathematical insights are used in a representative scientific practice.

Speaker: Michał Sochański (Adam Mickiewicz University in Poznań, Poland)

Title: Analogies between "mathematical experiments" and experiments in the natural sciences

Abstract: One of the most controversial and counter-intuitive points made about mathematical practice, is that some activities of mathematicians can be interpreted as "experiments". The latter term has been used in at least two ways: firstly, many mathematicians have reported performing "paper-and-pencil" experiments in their practice, such as calculating instances of a studied conjecture, trying out different manipulations with representation or testing new ideas; the second use of the term is connected with "computer experiments" understood as the use of dedicated mathematical software in investigating conjectures or in exploration (to name just two applications). In general, one can point at two characteristics of mathematical practices that have been considered to have "experimental" flavour: firstly, the similarity of those practices to procedures such as trial-and-error or testing new ideas, which are experimental on a common sense, non-scientific understanding of the term; secondly, the fact that they are – in some sense – analogous to experiments in the natural sciences. This second characteristic will be the focus of my talk.

The first question I will address is: how to make the analogy of certain mathematical practices with experiments in the natural sciences more explicit? Answering the question will involve finding analogons on the side of mathematical practice of such elements of physical experiments as: object and aim of the experiment, observation, evidence, experimental act, and finally result of the experiment and its interpretation. Those analogies will be analysed with respect to three main types of practices that have been called experimental: inductive procedures used to analyse instances of general statements, thought-experiments used to test new ideas, and experiments with representation. I will point at several types of analogies and argue that they are largely independent from each other. The discussion will also involve paying attention to differences between "paper-and-pencil" experiments and computer experiments.

The second question I will address is: what philosophical consequences can be drawn from the fact that a given analogy holds? Do they point at a *quasi*-empirical interpretation of mathematics or can they also be interpreted in the spirit of mathematical realism? Are there any philosophical consequences to be drawn at all? Answering those questions has to involve considering the crucial disanalogies between experiments in mathematics and in the natural sciences.

Speaker: Jean-Charles Pelland

Title: The discrete, the continuous, and the approximate number system

Abstract: It's a great time to be interested in numerical cognition. In the past few years, research involving adults, infants, and nonhuman animals has accumulated mountains of data supporting the existence of two separate cognitive systems - the Approximate Number System (ANS) and the Object-File System (OFS) - that appear to serve as building blocks for our formal arithmetical abilities. And yet, despite the widespread acceptance of the ANS as a legitimate explanandum of numerical behaviour, a handful of authors have recently questioned whether it is possible to create experimental conditions that can only be interpreted as evidence of the presence of innate cognitive systems with numerical content. Much of this controversy concerns whether the data are best explained by appealing to a system specifically tuned to detecting quantities of discrete items (the ANS), or whether it is more prudent to appeal to a more general analog magnitude system (the AMS). According to this latter interpretation of the data, our ability to respond to stimuli based on the number of items they contain would be due to the fact that number always co-varies with other magnitudes, such as size, luminosity, density, etc.

In this talk I sketch some of the main arguments levied against ANS-based theories and argue that on top of this empirical debate, there are conceptual considerations that suggest rejecting wholesale replacement of the ANS with an AMS. In a nutshell, even if it were possible to reinterpret

all numerical behaviour in terms of magnitude responses — and this is far from obvious — I claim that such an anumerical re-interpretation flies in the face of the extensive mathematical and philosophical literature concerning the relation between the continuous and the discrete. Proponents of the AMS seem to be ignoring arguments highlighting a conceptual opposition between the continuous and the discrete that prevents defining one in terms of the other that have been around since antiquity. To simplify the discussion, I focus for the most part on a recent critical review representative of this new wave of revisionist skepticism towards the ANS (Leibovich et al. 2017).

Session 2

Speaker: Ellen Lehet (University of Notre Dame, USA)
Title: The evolution of the group concept
Abstract: In this talk I will consider four different ways of conceptualizing groups: (1) in terms of transformations, symmetries, and rotations, (2) in terms of the standard axiomatic definition, (3) as groupoids, and (4) as group objects within a category. By considering these four characterizations of the group concept, we are able to see how mathematical methods have transformed into what they are today. In particular, we see how abstract, structural methods have become increasingly integral to the practice of mathematics. The discussion of this evolution and of the different perspectives on the concept of group will have two aims. The first aim is to highlight the epistemic value of structural methods within contemporary mathematics. The category theoretic perspective on groups in terms of groupoids and group objects introduces a level of structural abstractness that has proven to be advantageous in a variety of mathematical areas. For instance, the use of groupoids within algebraic topology has allowed for new approaches to and perspectives on theorems and proofs that are central to the field (e.g. the van Kampen theorem). This talk will aim to characterize the epistemic value of these structural methods. The second aim will be to bring to light some important features of mathematical progress. In particular, these new conceptualizations of the group concept are instances of mathematical progress and an understanding of what each of these conceptualizations contribute to the study of mathematics helps us to develop a complete understanding of the nature of mathematical progress. More specifically, the ways that each of these conceptualizations contribute to our mathematical understanding of the group concept indicate the significance of understanding for mathematical practice and progress. Overall, this discussion will shed light on the epistemic value of structural reasoning as well as illustrate the significance of this epistemic value for the progress of mathematics.

Speaker: Emmylou Haffner (Université Paris-Saclay, France)
Title: The shaping of rigorous mathematics, a case study with Dedekind’s drafts
Abstract: Many mathematicians’ testimonies suggest that the process of finding a new result or a new concept can be quite non-rigorous, the rigor being instilled later, as a subsequent step of the mathematical research. As a matter of fact, considerations about rigor in mathematics often rely on questions of justification and/or verification of results, rather than how they were found. In this talk, I will propose to shift the focus so as to look behind the scene and consider the shaping of rigorous mathematics. For this, I will use extracts from Richard Dedekind’s drafts in his Nachlass to analyze the genesis of some of his works, which are typically considered ‘rigorous’ 19th century mathematics.
After a presentation of Dedekind’s ideal of rigor based on statements in his published works, I will use drafts from his Nachlass to question the extent to which the research preliminary to the publication holds up to the same standards of rigor. I will show how the deductive hierarchy,

considered central to his ideal of rigor, is an aspect that emerges progressively and inductively, as what Dedekind considers the appropriate way to formulate definitions, theorems, etc., after analysing, generalizing, verifying mathematical experimentations and explorations. I will also consider whether the Dedekindian ideal of rigor guided mathematical research in its various phases, and what were consequences of such an ideal of rigor, if any, on mathematical research. I will suggest that rigor intervenes on two levels in these preliminary researches: regulating mathematical practice in the first steps of research, and more purposely in a step of rigorization that comes as a subsequent step in the writing process.

To do so, I will use two examples. Firstly, I will consider the genesis of his late Dualgruppe theory (equivalent to our modern lattice). Focusing on a specific law of Dualgruppe theory, I will show that the elaboration of a rigorous work can be the outcome of a process that is not necessarily so. I will put forward the trial-and-error and inductive aspects of Dedekind's research practices. Secondly, I will consider the genesis of Was sind und was sollen die Zahlen?, Dedekind's famous essay on the natural numbers. Dedekind wrote several versions of this text, from the 1870s to 1888. I will particularly be interested in what seems to be an important step of mathematical writing, in Dedekind's drafts, namely arranging the order of propositions in a deductive hierarchy.

Speaker: Elena Scalambro (University of Turin, Italy)

Title: On the mathematical practice of the Italian School of Algebraic Geometry: the case-study of Gino Fano

Abstract: While the features of the Italian School of Algebraic Geometry until the 1920s have been extensively studied, much less in known about the period 1923-1953. With the deaths of C. Segre (1924) and F. Klein (1925), leader and inspirer of the Italian School respectively, a certain kind of mathematical approach extinguished.

The aim of the present paper is to investigate how the geometrical practice changed and evolved in the three decades considered, starting from the case-study of Gino Fano (1871-1952), who was an outstanding mathematician and a major protagonist of this School. His figure is inextricably linked to the study and the classification of three-dimensional algebraic varieties, but he also gave some interesting – and, so far, not deeply analysed – philosophical contributions.

A first line of investigation involves the role of intuition in mathematical practice: indeed, while initially geometrical intuition was requested to overcome algebraic and analytical difficulties, which largely lay outside the possibilities of the time, then it gradually turned into a 'method'. As regards this aspect, Fano totally fitted into the Italian tradition: it also emerges from his attempt to extend the classical methods – which had been of extreme success for curves and surfaces – to varieties of higher dimension. However, this way of working by means of intuitive synthesis revealed its limits in the case of three-dimensional varieties.

Secondly, referring to the notion of 'mathematical style' as historiographical category according to Mancosu, Fano's works are emblematic of the Italian style in geometry: a synthetic approach is preferred, analogy and experimentation completely belong to the mathematical practice, results are often written with the goal of giving a 'perspective of their coming into being' and Italian leading scholars clearly think themselves as 'explorers of a new land'.

On one hand, Fano's late scientific production can be assumed as litmus test of the decline of the Italian geometrical research tradition and of the growing self-referentiality that affected Segre's School which, combined with a scenario of cultural autarchy, determined its decline. On the other hand, this case-study helps to highlight the inner-workings of mathematical communities and subcultures and contributes to show how the end-product of mathematical inquiry is inextricably tied to the production processes and to the context that constrains and sustains such mathematical construction.

Thursday July 1

Invited Speaker: Laura Crosilla (University of Oslo, Norway)

Title: Predicativity as invariance

Abstract: Predicativity emerged at the beginning of the 20th century within an exchange between Henri Poincare' and Bertrand Russell, following the discovery of the set-theoretic paradoxes. Hermann Weyl's fundamental book Das Kontinuum further shaped this notion. Predicativity may be seen as imposing a constructivity requirement on definitions of mathematical entities, which are to proceed as if the entities they define were constructed step-by-step and from below. Today, predicativity figures prominently in proof theory and in constructive set and type theories and, through the latter, also reaches the domain of constructive proof assistants. Notwithstanding its ubiquity, the very notion of predicativity requires clarification. A standard characterisation of predicativity takes impredicativity to involve vicious circularity. In this talk, I draw both on the early 20th century debate and on the contemporary mathematical practice to argue for another characterisation of predicativity that goes back to Poincare' and stresses a form of invariance of predicative definitions.

Session 1

Speaker: V.J.W. Coumans (Radboud University, The Netherlands)

Title: Definitions and concepts in mathematical practice

Abstract: Definitions are traditionally seen as abbreviations, as tools for notational convenience that do not increase inferential power. From a Philosophy of Mathematical Practice point of view, however, there is much more to definitions. For example, definitions can play a role in problem solving, definitions can contribute to understanding, sometimes equivalent definitions are appreciated differently, and so on.

This presentation concerns a research project that aims at understanding the various aspects of definitions in mathematical practice. One of the first observations is that *definitions* are intricately related to *concepts*. Some definitions are considered worthwhile because the concepts they introduce are relevant. Vice versa, sometimes there exists a concept of interest and a definition is valued because it helps the study of said concept. Consequently, to meaningfully discuss definitions in mathematical practice, one also needs to take the notion of concept into account.

In this presentation, I discuss the first two phases of this research project. Phase 1 constitutes a literature review of definitions and (to a certain extent) concepts in mathematical practice. I structured this literature review using four themes. These themes concern (1) the nature of definitions, (2) whether and how concepts evolve, (3) definitions and concepts from a communal perspective, and (4) different values relating to definitions and concepts.

Then, I will present preliminary results from the second phase of the project: an interview study. The aforementioned literature review serves as a basis for interviews with research mathematicians on how they perceive and interact with definitions and concepts. Together, the interviews and literature review shed light on how research mathematicians think about and interact with definitions and concepts.

Speaker: David Waszek (McGill University, Canada)

Title: Naturality of definitions and of notations

Abstract: As Jamie Tappenden (2008 a,b) has noted, contemporary mathematicians frequently discuss what the 'right' (or 'natural') definitions or concepts for a given area are. These discussions involve a subtle attitude to definitions. On the one hand, they are assessed on the basis of their ability to streamline current knowledge: for instance, a definition may be chosen because it allows stating general theorems, or writing down important proofs, in a (comparatively) simple way. On the other hand, definitions adopted on such grounds are often judged well-adapted to the 'nature of the subject' and are therefore used to guide further research.

One might think that such a methodological stance is typical of what historians sometimes call 'modern' mathematics (e.g., Benis-Sinaceur 2002; Gray 2008) or at any rate that it is no older than the 19th century. This paper, however, argues that a strikingly similar attitude can already be found in the late 17th century in Leibniz's work, and then traced into the 18th century, but in a form that concerns notations rather than definitions or concepts. Indeed, Leibniz, whose peculiar notational practices have already been noticed (e.g., Serfati 2008; Knobloch 2016), tended to adopt notations just because they allowed writing simple and general formulas, and in turn, to treat formulas that were simple and general (in his notations) as plausible—in a sort of reasoning that sometimes seems almost circular.

I shall focus on a clear case of such judgements of 'notational naturality', namely a particular notation introduced by Leibniz in 1694: the exponential notation for differentials (i.e., 'd^3x' instead of 'dddx', and so on). I shall review why Leibniz adopted it and how it shaped his, and Johann Bernoulli's, research and eventual discoveries. I shall then follow it into 18th century France, where it played an important role in the emergence of the so-called 'calculus of operations'. This phrase refers to a corpus of works by, among others, Lagrange, Laplace, Arbogast, and Servois, in which operators, for instance the 'd' of differentiation, are treated as if they were algebraic quantities whose 'powers' are interpreted as iterated applications of the operator (Koppelman 1971; Lubet 2010). My goal will be to explore the attitudes of these various authors, in particular Lagrange, towards the notation: to what extent and in what sense did they interpret its ability to summarise known results and yield novel ones as a sign of what we might call 'naturality'? How did they relate this 'naturality' to the possible existence of new mathematical objects, namely operators?

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Speaker: Walter Dean (University of Warwick, UK)

Title: Informal rigour and the origins of forcing

Abstract: This paper presents a conceptual reconstruction of the discovery of the method of forcing in set theory. The method is best known for its role in Paul Cohen's (1963; 1966) proof of the independence of the Continuum Hypothesis and Axiom of Choice from Zermelo-Fraenkel set theory. In this context, forcing is often presented (e.g. Cohen, 2002; Kanamori, 2008) as a technique which Cohen discovered largely in isolation from prior work in mathematical logic. Contrary to this understanding, my initial thesis will be that the development of forcing as a general method was the result of systematic reflection on concepts, results, and problems highlighted by work in hyperarithmetical theory and intuitionistic mathematics. Several of these were brought into focus by the "informally rigorous" examination of foundational standpoints associated with the work of Georg Kreisel in the 1950s-1960s.

In making this initial case, I will proceed as follows:

1) I will describe informal rigour as a means of analyzing concepts and addressing open questions based on the model developed in (Dean and Kurokawa, 2021).

2) A sketch of forcing will be presented highlighting two components:

a) A definition of a so-called forcing definition $\mathfrak{M}, Q \Vdash \varphi[Q/X]$ is introduced with the intended interpretation $\varphi(X)$ is determined to be true in the model $\mathfrak{M}$ when the variable X is interpreted as the set $Q \subseteq |\mathfrak{M}|$ by a finite amount of information about Q.

b) Presuming that $\mathfrak{M}$ already satisfies certain set existence axioms, an account is given as to why there also exists so-called generic sets Q which can be adjoined to its domain.

3) An account will be also given of how Kreisel was brought to introduce a forcing definition on the basis of his informally rigorous exploration of the concept of an extensionally indefinite property– a notion which arose in his foundational work on predicativism in (1960; 1961).

4) A parallel account will be given of how Kreisel was brought to consider the existence of generic sets on the basis of his informally rigorous exploration of the concept of an absolutely free (or lawless) sequence – a notion which arose in his foundational work on intuitionistic analysis in (1958; 1965; 1968).

5) On the basis of 3 and 4, I will reconstruct how Kreisel's analyses were brought together in the definition of arithmetical forcing by Feferman (1965) and discuss some of the initial results which were obtained on this basis in relation to Kreisel's foundational goals.

I will conclude by highlighting three senses in which these developments represent an important case study in the philosophy of mathematical practice:

i) They provide a paradigmatic illustration of how reflection on informal concepts can play a role in obtaining novel mathematical results.

ii) They illustrate how results which are often described as representing a discontinuous advance can be understood as the result of a conceptually motivated extension of prior work.

iii) They provide insight into how we might ultimately seek to understand Cohen's original independence results in from the standpoint of contemporaneous work in set theory rather than that of our latter-day technical understanding of forcing.

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Session 2

Speaker: Matteo De Benedetto (Ludwig-Maximilians-Universität, Germany)

Title: Conceptual Populations and Mathematical Selection: an evolutionary framework for conceptual change in mathematics

Abstract: The aim of this work is to propose a general evolutionary framework for conceptual change in mathematics compatible with the plurality of evolutionary dynamics exhibited by mathematical conceptual histories. I will build upon Mormann's (Mormann, 2002) selection theory for mathematical concepts and Godfrey-Smith's (Godfrey-Smith, 2009) population-based Darwinism. Godfrey-Smith's Darwinian framework is made of two ingredients: the family of concepts of a Darwinian population and several parameters tracking how much a certain population exhibits paradigmatic Darwinian features. I will mirror the coarsegrained structure of Godfrey-Smith's account, presenting a conceptual framework centered around the notion of a conceptual population, the opposition between Euclidean and Lakatosian populations, and the spatial tools of what I will call the Lakatosian space.

In my framework a conceptual population is composed by a set of conceptual variants and a set of mathematical problems together with a selection mechanism given by an heuristic power ordering of conceptual variants (relative to a given problem). I will present two ideals of conceptual populations, namely Lakatosian and Euclidean populations, that will represent (almost) opposite evolutionary dynamics. I will augment my framework with four parameters: conceptual variation, reproductive competition, environmental stability, and continuity. These four parameters that track how much a given conceptual population exhibits evolutionary features constitute the four dimensions of the Lakatosian space. Depending on how much they exhibit these parameters,

conceptual populations can be judged to be more Lakatosian or more Euclidean (or neither of them), occupying different regions of the Lakatosian space.

I will demonstrate how my framework, thanks to the four dimensions of the Lakatosian space, is able to give a fine-grained analysis of episodes of conceptual change in mathematics. I will furthermore show how my framework gives a novel perspective on whether conceptual change in mathematics is a rational process, distinguishing mathematical conceptual histories between cases of mathematical selection (global and local) and cases of evolutionary drift. I will do that by analyzing two case studies: Hamilton's invention of the quaternions (Hamilton, 1853), and the pre-abstract group concepts (Wussing, 1984).

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Speaker: Jan Makovský

Title: Foundations in service of education: calculus textbooks in 18th century Prague

Abstract: Despite the considerable development of the calculus in the 18th century and the early rise of foundational debates over the status of the infinitely small, the history of the calculus in Prague begins only in 1765 when Stepling's *Calculus differentialis* appeared. The first systematic textbook on the subject draws in a way from Euler's *Institutiones* but, at the same time, comes up as an original foundational attempt to explain the nature of the infinitely small in an arithmetical way. We shall devote the first part of the talk to the peculiarities (and failures) of Stepling's strategy as compared with the approaches of Euler, d'Alembert and others. While the concern of Stepling's work was logical in nature, the purpose of the other textbook we shall treat, Vydra's *Elementa calculi differentialis* (1783), was clearly limited to teaching: even though it was in fact a very simplified version of Stepling's *Calculus*. However, a major difference remains. Vydra, Stepling's disciple and successor, makes use of geometrical diagrams to illustrate Stepling's techniques deliberately freed of all geometrical reasoning. In the second part of the talk we will offer an analysis of the rather surprising teaching practice. In the end, we give some concluding remarks on the relations between foundational and educational issues about the calculus in 18th century Prague.

Speaker: Anna Kiel Steensen (ETH Zurich, Switzerland) Cancelled

Title: Textual practices of the formal: the case of lambda-calculus and combinatory logic

Abstract: This talk is about the syntax-semantics separation as a historical dynamic phenomenon. A common idea about the 'formal' is that it expresses a special relationship between a rule and its meaning. On the one side, we have 'syntax': rules and the process of following them blindly or automatically. On the other side is interpretation or 'semantics', which draws rules and their consequences into the world by relating them to ideas, practices, experiences, sensations, emotions, physical reality, etc. The 'formal', in this view, is that which operates independently of interpretation.

I claim that the historicity and materiality of the separation between syntax and semantics is not very well understood, and that we tend to consider the separation as an objective and transcendental relation. I would like to explore the opposite idea: that the 'formal' as operation without interpretation emerges *together* with the development of new mathematical, logical and computational practices.
I will focus on the theoretical and epistemic grounding of modern computer science in mathematics and logic by Alonzo Church and Haskell Curry, who in the middle of the 20th century sought to establish 'formal' notions of function and computation – precisely by transforming these deeply semantic notions into purely syntactic relations. This case study will explore how this work shaped our understanding of the 'formal'.
Because I aim to explore *how* syntax becomes separated from semantics, it cannot simply assume that the separation is a fact, which the texts merely express. Instead, the texts will be analyzed within a semiotic framework (building on structuralist approaches such as (Herreman 2000), (Hjelmslev 1943)), and the 'formal' will be described in terms of relations between the signs and practices emerging from the analysis of concrete texts.
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Friday July 2

Invited Speaker: Andrew Granville (Universite de Montréal, Canada)
Title: Cultural impact on the notion of incontrovertible proof in mathematics
Abstract: TBA

Session 1

Speaker: Ivo Pezlar (Czech Academy of Sciences, Czech Republic)
Title: What Speech Act Is Tied to a Contradiction?
Abstract: Recently, Ruffino, San Mauro, and Venturi (2020) argued in opposition to Ganesalingam (2013) that pragmatic phenomena (such as speech acts, implicatures, . . .) occur in the language of mathematics. Historically, probably the most well-known case of utilizing pragmatic notions in an otherwise rigorous language is Frege's vertical judgment stroke "|" that could turn propositions into judgments with assertoric force. The future development of predicate logic, to which Frege lay the foundations, however, abandoned this distinction and focused only on propositions. Although this dichotomy was lost for some time, it was not forgotten. When Martin-Löf started developing his type theory to provide the foundations for constructive mathematics, he made the distinction between judgments and propositions central to his framework. And while Frege had only one form of judgment, Martin-Löf introduced four.

In my talk I will consider other possible forms of judgments and speech acts, specially those that can be connected to the acts of reaching contradiction (absurdity)during derivations. Formally, contradiction “⊥” is most often treated as either a nullary logical connective or as a proposition that is always false. Tennant (1999) came with a competing proposal to view contradiction rather as a “structural punctuation mark.” Building on this approach I will explore the idea that contradiction is not just any punctuation mark but specially the exclamation point indicating a change of illocutionary force from declarative to imperative.

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Speaker: Peter Koepke (University of Bonn, Germany)

Title: The Language of Mathematics According to Ganesalingam and in the Naproche System

Abstract: "The Language of Mathematics" by Mohan Ganesalingam is an important standard reference for the actual mathematical language used in textbooks and papers. Ganesalingam's monograph is intended to apply to all of pure mathematics, following a Hilbertian axiomatic style where all notions and axioms are explicitly introduced. The book covers the interplay between natural language and symbolic material, using a bespoke type theory for disambiguation. The proposed semantics involves common foundational attitudes which lie between set theory and type theory.

Ganesalingam's analysis employs established techniques of formal and implementable linguistics, so that practical realizations of his approach, though ambitious, appear within reach. Indeed the preface of the book mentions a prototypical parser for mathematical language with a high success rate on actual textbook excerpts. Unfortunately this work has not appeared in public.

The Naproche Natural Proof Checking system accepts input in the controlled natural language ForTheL (Formula Theory Language) which is designed to approximate ordinary mathematical language and texts. It uses natural language processing for texts with symbolic material and strong automatic theorem proving for filling in implicit or obvious proof steps. Naproche allows the formalization and proof-checking of undergraduate mathematical texts in a style that is immediately readable by mathematicians. Example formalizations from various domains have been carried out.

The language processing of Naproche implements many of the principles identified by Ganesalingam: Naproche comes with a generic pattern-orientated phrase structure grammar which can also parse symbolic terms; the discourse representation theory of Ganesalingam's corresponds to parser and prover states; Naproche's inbuilt language is minimal and can be extended by new notions and axiomatic assumptions; the system uses a soft type theory and weak ontological assumptions on top of standard first-order logic.

It appears that further work can take ForTheL close to the "projected language" that Ganesalingam discusses in his doctoral thesis on which his monograph is based. This would have consequences for the analysis and future of mathematical practice. The technical core of mathematical texts would be captured by a controlled natural language with a clear formal semantics. Intuitive mathematical writing and fully formal texts are close together and sometimes even overlap, contrary to first impressions. There are some formalistic mathematical writings in the literature which are almost fully formal with respect to a rich controlled natural language. In a few years time, techniques of formal mathematics can be fitted with natural language interfaces and be intuitively used by mathematicians at large.

Speaker: Davide Rizza (University of East Anglia, UK)
Title: A framework for applications as problem-solving processes
Abstract: Recent philosophical discussions concerning the application of mathematics focus on the correspondence between empirical and mathematical structures (since Field (1980)) or on the issue of explanation (since Baker (2005)).
As a result, the analysis of applications has been persistently subjected to a counteproductive focus. In particular, the problem-solving character of applications has been concealed. Little attention has been paid to the fact that, in scientific enquiry, interrelated problems, rather than structured settings, present themselves first. Settings arise from after successful problem-solving techniques have been crystallised. Moreover, only after systematic work to bring problems under control has been carried out is it possible to consider certain facts as results of formal analysis, i.e. it is only after the construction of a problem-solving methodology by mathematical means that explanations arise as, possibly significant, byproducts.
My goal on this presentation is to refocus the study of applications around problem-solving and away from mirroring and explanation. I offer a general framework for the investigation of applications revolving around three distinctive phases. Then I apply this framework to a research programme from mathematical voting theory (see e.g. Saari (1994)), in order to show its usefulness.
My framework for the analysis of applications structures them around three phases, starting with an interrelated family of yet unsolved problems or a problematic field. Then:
(a) In the first phase, *a formal working environment is instated*. Terms from the problematic field are selected and their mathematical treatment is set up: in other words, their role in symbolic reasoning is specified as they are represented in a specific manner. I shall refer to the construction of their representation as the assignment of formal characters. The assignment of a formal character is what makes a term within a problem operative in reasoning.
(b) In the second phase, a formal working environment is instated and *a problem-solving methodology supported by the environment is constructed*. Its construction results in the systematic availability of selection and introduction operations. A selection operation singles out information overtly displayed by a problem, which, in the given formal environment, can be immediately processed in reasoning, e.g. inserted into a computation. An introduction operation identifies information not overtly displayed by the problem, which, in the given formal environment, can be immediately processed in reasoning and enables a later selection operation. A formal working environment that can bring a problematic field under control sustains a rich store of selection and introduction operations.
(c) In the third phase, the problem-solving methodology established in (b) undergoes expansion for the sake of assimilating new problems. As a result, the store of selection and introduction operations increases. In general, an expansion involves an enrichment of the formal working environment because it must integrate the terms of new problems.
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Session 2

Speaker: Paola Cantu (CNRS, Aix-Marseille Université, France)

Title: Philosophy of mathematical practice and social ontology

Abstract: There is a well-established tendency in recent philosophy of mathematics to emphasize the importance of scientific practice in answering certain epistemological questions such as visualization, the use of diagrams, reasoning, explanation, purity of proofs, concept formation, the analysis of definitions, and so on. Some of the approaches to mathematical practice investigate whether the objectivity of mathematical concepts is the result of a social constitution. For example, Salomon Feferman (2011) characterizes mathematical objectivity as a special case of intersubjective social objectivity. José Ferreiros (2016) defines mathematical practice as an activity supported by individual and social agents and characterized by stability, reliability, and intersubjectivity. Julian C. Cole (2013, 2015) sees mathematical objects as institutional rather than mental objects, explicitly referring to Searle's theory of collective intentionality and social ontology.

Even if tendencies in the direction of a disciplinary crossover between philosophy of mathematics and social ontology can be traced, no general survey has been offered yet. Applying two distinct approaches inspired respectively by the philosophy of mathematical practice and by social ontology to a specific case study, Peano's axiomatics, the paper will suggest further ways to compare and cross-fertilize the two philosophical approaches. Modern axiomatics can be investigated 1) as a mathematical method based on the distinction between axioms and theorems, rules of inference and definitions, or 2) as an institution based on rules, obligations, functions, coordination problems and agent's actions and roles. The example helps comparing the aims of the two approaches and the different kinds of problems they want to solve. But it also highlights complementarities between the respective conceptual tools.

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Speaker: Roy Wagner (ETH Zurich, Switzerland)

Title: Mathematical Consensus

Abstract: One of the distinguishing features of mathematics is the exceptional level of consensus among professional mathematicians. While this fact is brought up quite often, and some practical or metaphysical explanations have been suggested, I did not find in the literature sufficient analysis of what mathematicians actually agree on, how they achieve this agreement in practice, and under

what conditions. The talk is therefore meant as a programmatic intervention in the hope of motivating more research into this question.

It is commonplace to say that mathematicians agree on the validity of arguments (rather than, say, their importance, elegance, or notions of truth not reducible to provability). But even if we restrict attention to the validity of arguments, mathematicians sometimes find it hard to reach agreement. This issue persists even if we exclude the problem of context (e.g. textbook, classroom, levels of expertise) and focus on research communication between experts. In practice, agreement depends on a rather intricate "negotiation". Still, as some contemporary case studies suggest, there are cases where mathematicians fail to agree on the validity of some difficult proofs even after such process of negotiation.

To engage with this problematic, the talk will be divided into three parts.

In the first part, I will review the process of "negotiation" by which mathematicians achieve agreement about the validity of proofs. The process of isolating problematic aspects of a proof and revising them (by breaking them down into smaller components, translating them to a different "language" or modality, or arguing by analogy) most often does generate consensus. I will argue, however, that at this point a new kind of disagreement may arise: mathematicians may fail to agree whether the original proof and the re-negotiated proof are effectively the same or substantially different, and so may disagree whether the original proof is valid or not.

In the second part, I will briefly historicize the phenomenon of consensus. I will show that in earlier European mathematics (going as far as the early 20th centuries), consensus about the validity of arguments was substantially weaker than it is today, even if we exclude the famous foundations debates. Moreover, in various other mathematical cultures, the question of consensus had different forms and objectives. This means that contemporary consensus about the validity of mathematical proofs should be explained by recent historical changes in mathematical practice.

In the final part of the talk, I will try to explain what brought about the contemporary form of mathematical consensus. Since a sharp rise in consensus concerning the validity of proof occurs in the decades around the turn of the 20th century, it makes sense to explain this consensus by the concurrent logification and formalization of mathematics. However, this explanation has a major flaw: it explains a really existing phenomenon (consensus) by something that hardly ever happens (writing proofs in formal languages). I will therefore explain the ways in which aspects of formalization do enter mathematical practice so as to account for contemporary forms of mathematical consensus.

Speaker: Colin Jakob Rittberg (Vrije Universiteit Amsterdam, The Netherlands & Vrije Universiteit Brussel, Belguim)

Title: Intellectual Humility in Mathematics

Abstract: In this talk I explore how intellectual humility (fails to) manifest in mathematical practices. Some mathematicians claim that mathematical reasoning ensures humility because when Mr Nobody points out a flaw in Mrs Bigshot's proof, Mrs Bigshot has to concede. This seems to have broken down in the current debate about the abc conjecture; acclaimed mathematician Mochizuki holds on to his claim that his proof is correct, even after Fields medallist Scholze has pointed out what he considers to be a mistake in Mochizuki's proof. In this talk I analyse this disagreement amongst mathematicians in terms of intellectual humility. Virtue theorists are currently debating whether intellectual humility is best cashed out in terms of non-egotism or in terms of owning one's limitations. I employ both accounts in a case study of the abc conjecture disagreement to make their relative strengths and weaknesses visible. I draw particular attention to a symmetry in the abc conjecture disagreement: Mochizuki can charge the readers of his proof of not being sufficiently humble whilst his readers can charge him with the same blame. This symmetry is not discussed

amongst virtue epistemologists and hence a truly novel insight generated by a virtue-theoretic approach to the philosophy of mathematical practices. My talk furthermore challenges the Mr. Nobody / Mrs. Bigshot narrative by showing that despite the epistemic force of mathematical reasoning, in the here and now of mathematical activity there is room for disagreement about questions of validity and rigour even amongst some of the most prominent members of the mathematical community.

Saturday July 3

Invited Speaker: Orna Harari (Tel Aviv University, Israel)

Title: Finitism and Hero of Alexandria's Alternative Constructions: A Possible Explanation

Abstract: In his commentary on Euclid's Elements I.2, I.9, and I.12 Proclus presents three alternative constructions, which may go back to Hero of Alexandria, that address the objection that there is no place available to carry out the required construction. In my talk I attempt to understand the rationale behind this objection in light of a fragment from Alexander of Aphrodisias' commentary on Aristotle's Physics where he answers the question whether it is possible to construct an equilateral triangle on the dimeter of the cosmos in the negative. Arguing that this contention is not grounded in Alexander's understanding of mathematics and its objects but in his view that exceeding the limits of the universe is logically impossible, I suggest that the objection that Hero's contractions address is motivated by a similar concern. His definition of solids (Def. 11) may support this suggestion.

Session 1

Speaker: Vincenzo De Risi (Laboratoire SPHère, CNRS, France; and Max Planck Institute for the History of Science, Germany)

Title: Common Axioms in Euclid and Aristotle

Abstract: It is clear that there is a strong relation between Euclid's common notions in the *Elements* and the common axioms that Aristotle mentions in the *Metaphysics* and the *Posterior Analytics*. It seems very likely that some mathematicians of the age of Plato, such as Eudoxus of Cnidos, may have first introduced these principles. They were later adopted by Euclid in the *Elements* (which are a collection of previous mathematical treatises, including probably a few essays by Eudoxus himself), and discussed by Aristotle in his epistemological works. It is not likely that Euclid may have been directly influenced by Aristotle's work. This makes the relations between Euclid's and Aristotle's conceptions of common axioms even more interesting, since it shows how two very different personalities, a philosopher and a mathematician, may have read and interpreted the same set of principles.

In this talk, I argue that Euclid's first three common notions together provide a neat axiomatization of equality and additivity. While they may have first been conceived in geometry, they are easily and flawlessly generalized to numbers and magnitudes in general, thus fitting in with the Aristotelian remarks about κοινὰ ἀξιώματα. They are *entirely propositional* and their application does not rely on any diagrammatic inference. By contrast, the fourth and fifth common notions

listed in the *Elements* were external to this theory and responded to different epistemic needs. In particular, I claim that these common notions are employed in diagrammatic reasoning and display a different underlying epistemology than the other purely propositional principles.
I complement these results on ancient mathematics with a discussion on Aristotle's very different conception of the same axioms, and advance an "inferential" interpretation of Aristotle's views on axioms that is at odds with the standard reading of them as schemata of principles I conclude by showing the fruitful interplay between Aristotle's philosophical speculation and Euclid's mathematical practices.

Speaker: Benjamin Wilck

Title: Euclid's Measure of Complexity

Abstract: The sequences of Euclid's definitions form several well-ordered series such that Euclid systematically states the definitions of simpler geometrical objects before stating those of more complex ones. For instance, in Euclid's plane geometry, the circle is defined before the semicircle, the semicircle before the triangle, and the triangle before any of the quadrilateral figures. Thus, plane figures with a lesser are defined before those with a greater number of bounding lines, in which case these definitions follow the order of increasing complexity. This suggests that Euclid takes the complexity of a figure, in general, to be determined by the number of its external limits: the number of bounding lines determines the complexity of plane figures, while the complexity of solid figures will be determined by the number of bounding surfaces.
However, Euclid's definitions of solid figures contain two irregularities that seem to violate this order of increasing complexity. Firstly, Euclid's definitions of the pyramid and prism (XI.def.12–13) are stated before those of the sphere, cone, and cylinder (XI.def.14; 18; 21), although the sphere is bounded by 1 single surface, the cone by 2, and the cylinder by 3, whereas the pyramid is bounded by at least 4 surfaces and the prism by at least 5. Secondly, although the icosahedron has more faces than the dodecahedron, Euclid's definition of the dodecahedron (XI.def.28) is nonetheless stated after that of the icosahedron (XI.def.27).
Are these two sequences of definitions simply flawed, or why do they not conform to Euclid's otherwise systematic practice of defining figures with a lesser before those with a greater number of external limits? Does Euclid determine the complexity of a solid figure by something other than the number of its external surfaces? Can these tensions between Euclid's plane and solid geometries be resolved? By analyzing the sequences of the Elements' definitions in detail, this paper reconstructs Euclid's background assumptions pertaining to the measure of complexity of a geometrical object.

Speaker: Francesco A. Genco and Francesca Poggiolesi (CNRS, IHPST, Université Paris 1 Panthéon-Sorbonne, France)

Title: A Formal Analysis of Mathematical Explanations

Abstract: The idea that not every valid argument equally contributes to our understanding of its conclusion is almost as old as philosophy itself. Aristotle in the Posterior Analytics introduced a distinction between arguments that only certify the truth of their conclusion and arguments that also explain why their conclusion holds. He called the arguments of the second kind ἀποδείξεις (apodeixeis), that is, demonstrations. A particularly central aspect of these arguments—which is also rather hard to precisely frame—is that their premises should be, as Aristotle himself puts it, better known than their conclusion. Bolzano, Bohemian philosopher and mathematician that lived between the XVIII and the XIX century, took very seriously the endeavor of determining the characterizing features of explanatory arguments, and, for conceptual arguments of this kind,

proposed an analysis in terms of conceptual complexity: the arguments that display with utmost clarity why their conclusion holds are usually characterized by an increase in conceptual complexity from the premises to the conclusion. The premises contain, thus, simpler concepts.
In this work, we develop a formal framework for the characterization of the notion of mathematical explanation and for the analysis of mathematical explanations in terms of conceptual complexity. This endeavor presents two major difficulties. The first is due to the fact that developing an account of explanations in terms of conceptual complexity requires a method to analyze the structure of explanations. The method should, in particular, enable us to precisely evaluate the change in conceptual complexity induced by each step of the explanation. The second difficulty concerns the notion of conceptual complexity itself. Indeed, precisely framing the characteristics that make a concept more complex than another one is certainly not a negligible philosophical problem, and different concurrent criteria seem to be equally plausible.
Therefore, first of all, we present a formal method to structure mathematical explanations. In a nutshell the method consists into re-rewriting a mathematical explanation as a proof-tree where, in each node of the tree, one passes from some premises to some conclusion. Secondly, we discuss a formal characterization of the notion of conceptual complexity, which can be seen a refinement of the classical theory of concepts. According to this framework, a mathematical explanation is a proof-tree where conceptual complexity increases in each step of the tree.
We will show how this method works taking into account several different mathematical explanations, all in the realm of geometry: from some examples that can be found in Bolzano to Pythagoras' and Desargues' theorems.

Session 2

Speaker: Michael Friedman (Humboldt-Universität zu Berlin, Germany) Cancelled
Title: Joachim Jungius and the attempt to mathematize weaving
Abstract: Joachim Jungius (1587–1657), a German logician and mathematician is mostly well known for his *Logica Hamburgensis*. However, one of his hardly researched texts is a set of notes called *Texturae Contemplatio* from the 1630s and 1640s, containing reflections on textile practices, also accompanied with diagrams. In this manuscript, Jungius suggests to geometrize these practices, beginning the manuscript with numerous “definitions” and “theorems”, afterwards describing different weaving techniques and methods. While some parts of the text describe various practices (weaving or knitting, for example) as well as different materials, giving references either to religious or other ancient sources,[1] other parts certainly underline a mathematically influenced approach, supported visually by Jungius' diagrams. These notes point towards a possible mathematical theory of weaving patterns and practice, a theory which hardly existed in the 17th century. Moreover, books of diagrams for notating working with various loom were published in print only at the end of the 17th century–[2] underlining the importance of Jungius' manuscript.
The talk will examine in which ways Jungius' text attempted to (re)organize an existing system of knowledge, in order to constitute a more conceptual knowledge system. As I aim to show, Jungius aimed for a transformation of the artisanal practice of weaving and knitting into a more mathematical one – done by employing not only a Euclidean structure (as with his “definitions” and “theorems”), but also with various signs and diagrams, which were foreign to the practices artisanal activities.
Notes:
1. E.g. the old testament, Plato, Pliny the Elder.
2. See: Ziegler, Marx (1677), Weber Kunst und Bild Buch. Augsburg: Schultes; Lumscher, Nathanael (1708), Neu eingerichtetes Weber Kunst und Bild Buch. Bayreuth: Lumscher.

Speaker: Silvia De Toffoli (Princeton University, USA)

Title: Mathematical Practice and Analytic Epistemology

Abstract: Philosophers of mathematical practice have brought mathematics closer to philosophy of mathematics. This has led to a proliferation of new themes in philosophy of mathematics. These themes have to do with the epistemic role of visual imagination in mathematics, the nature of mathematical understanding, the purity of proofs, and the fact that some proofs are too big for a single agent to grasp, to name a few. Several of these new themes have to do with epistemology. Philosophers of mathematical practice, however, have not (yet) brought analytic epistemology closer to philosophy of mathematics. In my talk, I suggest that bringing together analytic epistemology and philosophy of mathematical practice would be advantageous for both fields. To be sure, there are some serious problems with the application of analytic epistemology to philosophy of mathematical practice. One is that adopting the traditional account of propositions in terms of possible worlds (à la Stalnaker), there is only one necessary proposition. To articulate an account of mathematical practice, we cannot work with this coarse-grained account of propositions. Relatedly, in analytic epistemology the center of the attention has been on highly-idealized subjects rather than historically-situated, interconnected agents. Moreover, when it has been considered at all, mathematics has generally been pictured by epistemologists as a realm of necessary truths, the knowledge of which can be gained a priori. Central questions in the epistemology of mathematics have been: how is it possible that mathematics is a priori? How can we know any mathematics? Moreover, it is generally assumed that all true mathematical propositions can be derived from first principles through truth-preserving deductions. This assumption has led to disregarding the question of how individual beliefs are justified to focus on the problem of how to justify mathematical theories. I explore ways in which to overcome these problems. In particular, I refer to specific trends in analytic epistemology, viz. social epistemology and feminist epistemology. I then discuss several themes of analytic epistemology that could be applied to mathematical practice. Among these are the relationship between knowledge-that and knowledge-how, group knowledge and justification, the epistemic significance of disagreement and higher-order evidence more generally, and the relationship between propositional and doxastic justification. I suggest that not only it would be beneficial to analytic epistemology to use mathematical practice as a testing ground for new theories, but also that philosophy of mathematical practice would find a variety of useful items in the toolbox of analytic epistemology.

Speaker: Guillermo Nigro (UFBA, Brazil & UdelaR, Uruguay))

Title: Purity of methods and mathematical practice. A discussion of the abstract approach of Detlefsen, Arana and Mancosu

Abstract: In a series of articles, alone or in collaboration, Andrew Arana, Michael Detlefsen (1948 - 2019), and Paolo Mancosu have proposed and defended a notion of *purity of methods* (or *pure solution*). Intuitively, "pure" methods are somehow "intrinsic" to the problem or theorem in whose solution/proof they employ. On the other hand, a methodological resource is "impure" if it turns out to be "foreign" to the problem. Since "intrinsic" and "foreign" do understand it as a *topical (semantic)* relation, the authors refer to it as "topical purity". Thus, topically pure solutions to problems draw only on what is – semantically– constitutive of that problem's identity. A classic example, particularly important in 19th Century's geometry, is the belief that in study properties of geometrical figures, the uses of coordinates' systems is "foreign" or, in Chasles' words: "auxiliaire et artificiel" (Chasles, 1989, p. 119).

We can call the above way of introducing the problem "abstract," and the reason for doing so is the following: it consists in introducing, informally, a relation – "intrinsic"–between the topic of the theorem/problem and the topic of the proof/solution, to be rigorously characterized. To do this, we can introduce (define) a concept of "content," or "topic" of a problem/theorem, first, and then, to define –regarding that concept– the relation "intrinsic," based on some "adequacy conditions" for it. These adequacy conditions must offer us a way to distinguish between what is "intrinsic" to the problem's content and what is "foreign" to it.

Thus, Detlefsen and Arana (2011), and Arana and Mancosu (2012) presents and defends two different but complementary concepts of content: "basic understanding" and "informal content" ("ordinary understanding"), respectively. Broadly, these concepts consider what is *explicitly stated* in problems' or theorems' formulations and assign to mathematical expressions involved in them the most "basic," or "ordinary" definitions. Therefore, we can reformulate the notion of "topical pure solution:" topically pure solutions to problems draw only on what is constitutive of the *basic understanding*, or *informal content* of that problem. Thus, these notions of "basicness" or "ordinariness" are essential for Detlefsen, Arana, and Mancosu's conception of purity pf methods.

Both the cases of logical consequence and the cases of pure solutions are *products* of mathematical practice, and the respective abstract concepts do not inform us about the *methodological choices behind them*.[1] In this sense, then, we can call this way of introducing the problem of method purity "abstract".

In this talk I suggest, on the other hand, it suggests that Detlefsen, Arana and Mancosu's approach is neither informed by nor informs us by mathematical practice. This situation is due to the theoretical approach's abstract character, especially to the notion of "content" or "topic" with which they are engaging to offer a discussion of Detlefsen, Arana and Mancosu's approach.

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Notes:

1. Compare the *philosohical* distinction between "history" and "heritage" in Grattan-Guiness (2004, p. 16): heritage tends to focus upon knowledge alone (theorems as such, and so on), while history also seeks causes and understanding in a more general sense.